Strong Symplectic Fillings of Contact Manifolds through Lefschetz-Bott Fibrations: Revisited

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Abstract

We review two theorems of Oba [Oba20], concerning the existence of a symplectic Lefschetz-Bott fibration on a complex line bundle over a symplectic manifold with a Donaldson hypersurface, and the application thereof to the link of the A_k-type singularity, obtaining distinct strong symplectic fillings of the link. To this end, we first provide the necessary background on symplectic fillings, Lefschetz and Lefschetz-Bott fibrations, and open book decompositions.

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Chapter 0

Introduction

The various notions of *symplectic fillings* provide a tool to study the topology of contact manifolds. Generally speaking, a contact manifold (M, ξ) is symplectically fillable if it can be realized as the boundary of a symplectic manifold (W, ω) in such a way that the symplectic form ω on W is compatible in a suitable sense with the contact structure ξ on M, and there are various increasingly restrictive conditions one may impose on (W, ω) to give rise to different flavours of fillings.

It is known that not every contact manifold is symplectically fillable - indeed, every symplectically fillable contact structure is necessarily tight [Eli91], which presents an effective tool to prove tightness of contact structures aside from holomorphic curve theory introduced by Gromov [Gro85].

Another fruitful tool in the study of the topology of a manifold is presented by open books, decomposing the manifold of interest into codimension one *pages*, revolving around a *binding*. The first use of open books was in a theorem of Alexander [Ale23], establishing that any topological 3-manifold admits an open book description. In the literature, the underlying structure of open books appears under various names in diverse contexts, such as *relative mapping tori*, *fibered links*, or *spinnable structures* [Tam72], and Milnor's Fibration Theorem treated what would in modern terminology be known as open book decompositions of spheres [Mil68].

It was in 1973 that Winkelnkemper [Win73] coined the term *open book*, and in collaboration with with Thurston, they explained in [TW75] how to endow any open book of a 3-manifold with a contact structure using Alexander's theorem.

Alexander's theorem was generalized to closed manifolds of odd dimension greater or equal to 7 by Winkelnkemper, Tamura [Tam73], and Lawson [Law78] between 1973 and 1978. Quinn [Qui79] further extended this result to closed manifolds of dimension at least 5 in 1979, establishing that any odd-dimensional closed manifold admits an open book decomposition.

Giroux and Mohsen generalized Thurston and Winkelnkemper's construction to arbitrary odd dimensions when the pages of the open book are Liouville domains [GM]. Hence, open books give rise to contact manifolds, and it is shown in [GM] that, in fact, any contact manifold admits a supporting open book. In dimension three, Giroux discovered that the correspondence between open books and contact structures up to isotopy on a given manifold is in fact unique up to an equivalence relation called *positive stabilization* [Gir02].

The versatility of open books reveals itself in conjunction with symplectic Lefschetz fibrations as a means to obtain Stein fillings of contact manifolds. A complex analogue of Morse functions introduced by Donaldson [Don99] in the context of symplectic geometry, restricting a symplectic Lefschetz fibration $\pi : (E, \Omega) \to \mathbb{D}$ over the unit disk $\mathbb{D} \subset \mathbb{C}$ to the boundary ∂E induces a contact open book description of ∂E, producing a contact manifold.

By Eliashberg's characterization of Stein domains [Eli91], the total space E can be seen to be a Stein filling of this contact manifold, which makes it possible in certain cases to read off fillability of a contact manifold directly from a contact open book description. In fact, a converse was given by Giroux and Pardon [GP17], establishing that any Stein domain can be presented as the total space of a symplectic Lefschetz fibration over the disk.

The technique of filling by Lefschetz fibrations has been applied by Özbağcı and Stipsicz to construct 3-manifolds with infinitely many Stein fillings [ÖS04a], and Oba has generalized this result to higher dimensions [Oba18].

By allowing the critical locus of the fibration π to be a submanifold rather than a discrete set of points, we generalize Lefschetz fibrations to the notion of *Lefschetz-Bott fibrations*. Formally studied by Perutz in the construction of Lagrangian matching invariants [Per07], restricting a symplectic Lefschetz-Bott fibration to its boundary again produces a contact manifold for which its total space serves as a strong symplectic filling [Oba20], [LHW18]. Notably, fillings induced by symplectic Lefschetz-Bott fibrations need not be Stein in general (see Remark 4.7.4).

The main purpose of this text is to examine how Oba in [Oba20] has established the existence of symplectic Lefschetz-Bott fibrations on line bundles over a class of symplectic manifolds, and how they can be applied to obtain distinct strong symplectic fillings of the link of the A_k-type singularity.

To this end, we organize this thesis as follows. Chapter 1 to Chapter 3 serve as preparation for the main applications: in Chapter 1, we review the fundamentals of symplectic and contact geometry before introducing the terminology of symplectic fillings. Chapter 2 consists of a discussion of Lefschetz and Lefschetz-Bott fibrations, and Chapter 3 reviews the theory of open books before exploring how to obtain a filling of a contact manifold through Lefschetz and Lefschetz-Bott fibrations.

Having established the necessary background, Chapter 4 is concerned with the proof of [Oba20, Theorem 1.1], which guarantees the existence of a symplectic Lefschetz-Bott fibration on a complex line bundle over a polarized symplectic manifold. In Chapter 5, we explain how to obtain distinct symplectic fillings of the link of the A_k-type singularity using symplectic Lefschetz-Bott fibrations, which amounts to the proof of [Oba20, Theorem 1.2].

We conclude in Chapter 6 by indicating leads as to how one might be able to prove similar filling results for other contact manifolds, and by exploring what objects involved in the construction of the Lefschetz-Bott fibration from [Oba20, Theorem 1.1] would need to be better understood in order to make them more explicit in a simple case.

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Chapter 1

Symplectic Fillings

The setting throughout this thesis is that of symplectic geometry, so we start this chapter by recalling the relevant definitions. Following up in Section 1.1.2, we will introduce some notions of contact geometry, which can be considered as the geometry occuring on hypersurfaces contained in a neighbourhood where the symplectic form is exact.

Having set the stage, we will introduce symplectic fillings of contact manifolds, one of the main points of interest in this thesis.

1.1 Setting the Stage

1.1.1 Symplectic Geometry

Definition 1.1.1. A **symplectic manifold** is a pair (W, ω) where W is a manifold equipped with a 2-form $\omega \in \Omega^2(W)$ that is

- closed, i.e. , $d\omega = 0$, where d denotes the exterior derivative;
- nondegenerate, by which we mean that for any point $x \in W$ and any nonzero tangent vector $u \in T_xW$, the map

$$\begin{aligned} T_xW &\to T_x^*W \\ u &\mapsto \omega_x(u, v) \end{aligned}$$

  is an isomorphism.

A symplectic manifold (W, ω) is called **exact** if ω is exact.

Remark 1.1.2. The nondegeneracy condition on the symplectic form ω is equivalent to ω^n, the top exterior power of ω, being a volume form. This implies that the dimension of W is even, since the top power of ω will always have even degree as ω is a 2-form.

Moreover, there are three distinguished classes of submanifolds of symplectic manifolds we may occasionally reference.

Definition 1.1.3. Let (W,ω) be a symplectic manifold and $L \subset W$ an embedded submanifold. For any point $x \in L$, denote by

$$(T_xL)^\omega := \{v \in T_xM \mid \omega_x(v,w) = 0 \, \forall w \in T_xL\}$$

the **symplectic complement** of T_xL. The submanifold L is called

- **isotropic**, if $(T_xL) \subset (T_xL)^\omega$ for all $x \in L$;
- **coisotropic**, if $(T_xL)^\omega \subset T_xL$ for all $x \in L$;
- **Lagrangian**, if $T_xL = (T_xL)^\omega$ for all $x \in L$.

Remark 1.1.4. Note that L being isotropic is equivalent to the symplectic form ω vanishing when restricted to TL. Moreover, if $\dim W = 2n$, it is easy to prove that the dimension of an isotropic submanifold is at most n, whereas coisotropic submanifolds are of dimension at most n. Consequently, Lagrangian submanifolds have dimension n.

Some prototypical examples of symplectic manifolds which will appear throughout the rest of this thesis are the following:

Examples 1.1.5.

1. Let $W = \mathbb{R}^{2n}$ with linear coordinates $x_1, \ldots, x_n$, $y_1, \ldots, y_n$. Then the standard symplectic structure on $\mathbb{R}^{2n}$ is

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

   One may easily check that ω_0 is closed and nondegenerate.

2. Let $W = \mathbb{C}^n$ with complex linear coordinates $z_1, \ldots, z_n$. Identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ via $z_j = x_j + iy_j$, and define the 1-forms $dz_j = dx_j + idy_j$ and $d\bar{z}_j = dx_j - idy_j$, for $j = 1, \ldots, n$.

   The standard symplectic form on $\mathbb{C}^n$ is

$$\omega_0 = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j.$$

   Note that this is precisely the standard symplectic form on $\mathbb{R}^{2n}$ under the identification $z_j = x_j + iy_j$.

3. Complex projective space $\mathbb{C}P^n$ carries a symplectic structure, which can be characterized as follows.

   Let $p : S^{2n+1} \to \mathbb{C}P^n$ be the Hopf fibration and $i : S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$ the inclusion. Then the **Fubini-Study form** $\omega_{\mathrm{FS}} \in \Omega^2(\mathbb{C}P^n)$ is the unique symplectic form satisfying $i^*\omega_0 = p^*\omega_{\mathrm{FS}}$.

Recall that the standard atlas is made up of charts of the form $(\varphi_j, \mathcal{U}_j)$, where $\mathcal{U}_j = \{[z_0 : \ldots : z_n] \in \mathbb{C}P^n \mid z_j \neq 0\}$, and

$$\varphi_j([z_0 : \ldots : z_n]) = \frac{1}{z_j}(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n).$$

It can be shown that the map

$$\begin{aligned} \mathbb{C}^n &\to \mathbb{R} \\ \mathbf{z} &\mapsto \log(|\mathbf{z}|^2 + 1) \end{aligned}$$

is i-convex, so that the 2-form $\tilde{\omega}_{FS} := -dd^{\mathbb{C}} \log(|\mathbf{z}|^2 + 1)$ defines a symplectic form on $\mathbb{C}^n$. The transition functions of the above atlas for $\mathbb{C}P^n$ preserve $\tilde{\omega}_{FS}$, so that one may pull $\tilde{\omega}_{FS}$ back by the maps φ_i to obtain a well-defined symplectic form on $\mathbb{C}P^n$. This symplectic form turns out to coincide with ω_{FS}. See Section 1.4.1 for definitions relating to i-convexity, and [Can06, Chapter 16] for more details on the Fubini-Study form ω_{FS}.

The notion of equivalence for symplectic manifolds is that of *symplectomorphisms*.

Definition 1.1.6. A **symplectomorphism** φ between two symplectic manifolds (W_1, ω_1) and (W_2, ω_2) is a diffeomorphism $\varphi : W_1 \to W_2$ such that $\varphi^* \omega_2 = \omega_1$.

One of the first statements one typically encounters in the study of symplectic geometry is that symplectic manifolds "have no local invariants": locally, every symplectic manifold looks like $\mathbb{R}^{2n}$ with the symplectic structure ω_0 from Examples 1.1.5. The formal statement is given by Darboux' theorem.

Theorem 1.1.7 (Darboux)**.** *Let (W^{2n}, ω) be a symplectic manifold of dimension $2n$. Then for every point $x \in W$, there exists a neighbourhood $U \subset W$ of x, a neighbourhood V of $\mathbf{0} \in \mathbb{R}^{2n}$, and a chart $\varphi : U \to V$ so that $\varphi(x) = \mathbf{0}$, and*

$$(\varphi^{-1})^* \omega = \omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is the standard symplectic structure on $V \subset \mathbb{R}^{2n}$.

For a proof, see [Can06, Theorem 8.7].

1.1.2 Contact Geometry

A contact structure on an odd-dimensional manifold M is a codimension one distribution, subject to a non-integrability condition. Often referred to as the odd-dimensional cousin of symplectic geometry, the two fields are closely linked and exhibit some similarities. The exposition of contact geometry in this thesis follows that of Geiges [Gei08].

To describe codimension one distributions, it is useful to consider them as the kernel of 1-forms. This is always possible locally, and often enough, there is a global 1-form α defining ξ:

Lemma 1.1.8 ([Gei08, Lemma 1.1.1])**.** *Let ξ be a codimension one distribution on a manifold M. Then ξ can locally be written as the kernel of a 1-form α. It is possible to write $\xi = \ker \alpha$ for a global 1-form α if and only if ξ is* ***coorientable****, which is to say that the quotient line bundle TM/ξ is trivial.*

For the rest of this thesis, we shall assume all our hyperplane fields to be coorientable unless otherwise specified.

Recall that a distribution can be integrated to a foliation if the set of vector fields belonging to ξ form a subalgebra of the Lie algebra of vector fields under the Lie bracket. One can show that, in terms of the defining 1-form α, this is equivalent to

$$\alpha \wedge d\alpha \equiv 0.$$

This particular result follows from [Tam76, The Frobenius Theorem 7.10]. The maximal non-integrability criterion which makes a hyperplane distribution into a contact structure reads as follows:

Definition 1.1.9. Let M be a manifold of dimension $2n + 1$. A **contact structure** ξ on M is a codimension one distribution $\xi = \ker \alpha$ such that

$$\alpha \wedge (d\alpha)^n \neq 0.$$

The 1-form α is called a **contact form**, and the pair (M, ξ) is then called a **contact manifold**.

Remark 1.1.10. As $\alpha \wedge (d\alpha)^n$ is a volume form, contact manifolds are in particular orientable. Given an orientation of M, a contact form α is called **positive** if the orientation induced by $\alpha \wedge (d\alpha)^n$ agrees with the one prescribed, and **negative** otherwise.

Remark 1.1.11. An equivalent characterisation of the contact condition for a 1-form α is that $d\alpha$ is symplectic on ξ.

To see this, let α be a contact form and choose any vectors $u_0, \ldots, u_{2n}$ so that

$$\alpha \wedge (d\alpha)^n(u_0, \ldots, u_{2n}) \neq 0.$$

As M is $(2n+1)$-dimensional and $\ker \alpha$ is $2n$-dimensional, precisely one of the vectors must not lie in $\ker \alpha = \xi$. Without loss of generality, let this vector be u_0. This implies $(d\alpha)^n(u_1, \ldots, u_{2n}) \neq 0$, so $(d\alpha)^n$ is a volume form on ξ.

If $d\alpha$ is nondegenerate on ξ, we may choose $u_1, \ldots, u_{2n} \in \xi$ so that

$$(d\alpha)^n(u_1, \ldots, u_{2n}) \neq 0.$$

Again by dimensional reasons, there must exist a vector $u_0 \notin \xi$ so that $(u_0, u_1, \ldots, u_{2n})$ is linearly independent, and also

$$\alpha \wedge (d\alpha)^n (u_0, u_1, \ldots, u_{2n}) \neq 0.$$

To a contact form, we associate a unique vector field as follows:

Definition 1.1.12 ([Gei08, Lemma 1.1.9])**.** Let α be a contact form defining a contact structure $\xi = \ker \alpha$. Then there exists a unique vector field R_α, called the **Reeb vector field** of α, satisfying

- $\imath_{R_\alpha} d\alpha = 0$
- $\alpha(R_\alpha) = 1$.

Note that while R_α is unique, there is no well-defined Reeb vector field associated to a contact *structure*, as there are many 1-forms having ξ as their kernel.

The following examples of contact structures will occasionally appear throughout this text:

Example 1.1.13.

1. The standard contact structure ξ_0 on $\mathbb{R}^{2n+1}$ with coordinates

   $$(x_1, y_1, \ldots, x_n, y_n, z)$$

   is defined as the kernel of the 1-form

   $$\alpha_0 := dz + \sum_{i=1}^{n} x_i dy_i.$$

   This is indeed a contact form as $d\alpha_0 = \sum_{i=1}^n dx_i \wedge dy_i$ is the standard symplectic form on $\mathbb{R}^{2n}$.

2. As we will soon see in Definition 1.2.2, one way to obtain contact forms is through so-called *Liouville vector fields* on symplectic manifolds (W, ω). These are vector fields expanding the symplectic form in the sense that

   $$\mathcal{L}_V \omega = \omega.$$

   It turns out that if V is a Liouville vector field transverse to some hypersurface $\Sigma \in W$, then $\imath_V \omega$ restricted to Σ is a contact form.

   We use this notion to define the standard contact structure ξ_{can} on S^{2n+1}. Consider $\mathbb{R}^{2n+2}$ with its standard symplectic form $\omega_0 = \sum_i dx_i \wedge dy_i$. The (slightly scaled) radial vector field

   $$V(\mathbf{x}, \mathbf{y}) := \frac{1}{2} \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}$$

is Liouville for ω_0: we have

$$\lambda_0 := \imath_V \omega_0 = \frac{1}{2}\sum_{i=1}^{n+1}(dx_i(V)dy_i - dy_i(V)dx_i) = \frac{1}{2}\sum_{i=1}^{n+1}(x_i dy_i - y_i dx_i).$$

λ_0 can easily be checked to be a primitive of ω_0, so that $\mathcal{L}_V\omega_0 = \omega_0$.

Evidently, V is outward pointing on S^{2n+1}, so that λ_0 is a contact form on S^{2n+1}. Fix $\xi_{\text{can}} = \ker \lambda_0$ as the standard contact structure on S^{2n+1}.

Equivalence among contact manifolds is described by *contactomorphisms*.

Definition 1.1.14. A **contactomorphism** between contact manifolds (M_1, ξ_2) and (M_2, ξ_2) is a diffeomorphism $\varphi : M_1 \to M_2$ such that $D\varphi[\xi_1] = \xi_2$.

Remark 1.1.15. The condition for φ to be a contactomorphism in terms of contact forms is that $\varphi^*\alpha_1$ and α_0 have the same kernels, which is the case if and only if $\varphi^*\alpha_1 = f\alpha_0$ for some nowhere zero function f.

It is at this point that a first similarity to symplectic geometry arises in that both geometries have no local invariants.

Theorem 1.1.16 (Pfaff)**.** *Let α be a contact form on the manifold M^{2n+1} and $x \in M$ be a point. Then there exists a neighbourhood $U \in M$ of x, a neighbourhood V of $\mathbf{0} \in \mathbb{R}^{2n+1}$, and a chart $\varphi : U \to V$ so that $\varphi(x) = \mathbf{0}$, and*

$$(\varphi^{-1})^*\alpha = \alpha_0 = dz + \sum_{i=1}^{n} x_i dy_i.$$

See for example [Gei08, Theorem 2.5.1] for a proof.

Let us examine the Reeb vector field of λ_0. Because general dimension $(2n+1)$ only increases notational complexity, in what follows we will consider the case of S^3.

Example 1.1.17. The Reeb vector field R_{λ_0} for the contact form

$$\lambda_0 = \frac{1}{2}\left(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2\right)$$

on S^3 is

$$R_{\lambda_0} = 2\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2}\right).$$

Moreover, the orbits of its flow define the fibers of the **Hopf fibration**

$$\begin{aligned} \mathbb{C}^2 \supset S^3 &\to S^2 = \mathbb{C}P^1, \\ (z_1, z_2) &\mapsto [z_1 : z_2]. \end{aligned}$$

Proof. This proof is as in [Gei08, Lemma 1.4.9].

We will denote the proposed candidate for R_{λ_0} by Z and show that $Z = R_{\lambda_0}$. Then

$$\imath_Z \lambda_0 = x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1,$$

and as $d\lambda_0 = \omega_0$, we see that

$$\begin{aligned}\imath_Z \omega_0 &= 2(-y_1 dy_1 - x_1 dx_1 - y_2 dy_2 - x_2 dx_2)\\ &= -2r dr.\end{aligned}$$

However, on the tangent bundle of S^3, $dr \equiv 0$ since $r \equiv 1$. This proves $Z = R_{\lambda_0}$.

The fibers of the Hopf fibration containing the point

$$(z_1, z_2) = (x_1, +iy_1, x_2 + iy_2) \in S^3 \subset \mathbb{C}^2$$

can be parametrised by

$$\gamma(t) = (e^{it} z_1, e^{it} z_2), \quad t \in [0, 2\pi)\,.$$

We claim this is an integral curve of R_{λ_0}. Evidently $\dot\gamma(t) = (ie^{it} z_1, ie^{it} z_2)$, which in real coordinates is

$$\begin{aligned}&(x_1\cos(t) - y_1\sin(t))\frac{\partial}{\partial y_1} - (y_1\cos(t) + x_1\sin(t))\frac{\partial}{\partial x_1}\\ +&(x_2\cos(t) - y_2\sin(t))\frac{\partial}{\partial y_2} - (y_2\cos(t) + x_2\sin(t))\frac{\partial}{\partial x_2}.\end{aligned}$$

This is easily checked to be $\frac{1}{2}R_\alpha(\gamma(t))$, which yields the claim. Note that in particular, the orbits of R_{λ_0} are closed. □

1.2 Symplectic Collar Neighbourhoods

As remarked before, contact geometry naturally arises on the boundary of symplectic manifolds, under the assumption that there exists a transverse Liouville vector field near the boundary. The goal of this subsection is to show that a neighbourhood of the boundary of such symplectic manifolds can be *symplectized*, and this neighbourhood will be referred to as a *symplectic collar*. Symplectic collar neighbourhoods will be ubiquitous in local computations in the chapters to follow.

Definition 1.2.1. Given a contact manifold $(M^n, \xi = \ker\alpha)$, the **symplectization** of (M, ξ) is the symplectic manifold

$$(\mathbb{R} \times M, d(e^t\alpha)),$$

where we identify α with its pullback under the projection $\mathbb{R} \times M \to M$.

This is indeed a symplectic form:

$$(d(e^t\alpha))^n = (e^t(dt \wedge \alpha + d\alpha))^n = ne^{nt} dt \wedge \alpha \wedge (d\alpha)^{n-1} \neq 0.$$

The assumption mentioned above was the existence of a *Liouville vector field*:

Definition 1.2.2. Let (W^{2n}, ω) be a symplectic manifold. A vector field V on W is called **Liouville** if

$$\mathcal{L}_V \omega = \omega.$$

Remark 1.2.3. One should note the following consequences of this definition.

1. Wherever V is defined, ω is exact: we have $\omega = \mathcal{L}_V\omega = d\imath_V\omega$, so that $\omega = d\lambda$, where $\lambda := \imath_V\omega$.
2. Conversely, if ω is exact, then any primitive λ induces a Liouville vector field defined by $\lambda = \imath_V\omega$, which exists by nondegeneracy of ω.
3. If $\Sigma \subset W$ is any orientable hypersurface transverse to V, then $\lambda|_{T\Sigma}$ is a contact form on Σ: as V is transverse to Σ, $\imath_V(\omega^n)$ is a volume form on Σ, so that
$$0 \neq \imath_V(\omega^n) = n\imath_v\omega \wedge \omega^{n-1} = n\lambda \wedge (d\lambda)^{n-1}.$$
Note that in the case $\Sigma = \partial W$, the contact structure induced by $\lambda|_{T\partial W}$ is positive or negative depending on whether V is inward or outward pointing, respectively.

Remark 1.2.4. A hypersurface which has a neighbourhood in which a Liouville vector field is defined is thus often referred to as a hypersurface of **contact type**. We will primarily consider $\Sigma = \partial W$.

Relating this to the symplectization of a contact manifold, we have the following:

Proposition 1.2.5. *The vector field $\frac{\partial}{\partial t}$ is a Liouville vector field on the symplectization of the contact manifold $(M, \xi = \ker\alpha)$.*

Proof. We need to show $\mathcal{L}_{\frac{\partial}{\partial t}} d(e^t\alpha) = d\imath_{\frac{\partial}{\partial t}} d(e^t\alpha) \overset{!}{=} d(e^t\alpha)$. Therefore, it suffices to show that $\imath_{\frac{\partial}{\partial t}} d(e^t\alpha) = e^t\alpha + \beta$ for some closed 1-form $\beta \in \Omega^1(\mathbb{R}\times M)$. Take any $Y \in \mathfrak{X}(\mathbb{R}\times M)$ and compute

$$\begin{aligned}\imath_{\frac{\partial}{\partial t}} d(e^t\alpha)(Y) &= e^t(dt\wedge\alpha + d\alpha)\left(\frac{\partial}{\partial t}, Y\right)\\ &= e^t\left(\alpha(Y) + d\alpha\left(\frac{\partial}{\partial t}, Y\right)\right).\end{aligned}$$

Recall that we are writing α for $\mathrm{pr}^*\alpha$ by abuse of notation, where $\mathrm{pr} : \mathbb{R}\times M \to M$ is the projection. We shall write this explicitly to show that $d\alpha\left(\frac{\partial}{\partial t}, Y\right)$ vanishes. Write $Y = f\frac{\partial}{\partial t} + g^i\frac{\partial}{\partial x_i}$ for some smooth functions $f, g^i \in C^\infty(\mathbb{R}\times M)$

and local coordinates x_i on M and compute

$$
\begin{aligned}
d\mathrm{pr}^*\alpha\left(\frac{\partial}{\partial t}, Y\right) &= \frac{\partial}{\partial t}(\mathrm{pr}^*\alpha(Y)) - Y\left(\mathrm{pr}^*\alpha\left(\frac{\partial}{\partial t}\right)\right) - \mathrm{pr}^*\alpha\left(\left[\frac{\partial}{\partial t}, Y\right]\right) \\
&\overset{(1)}{=} \frac{\partial}{\partial t}\left(\mathrm{pr}^*\alpha\left(f\frac{\partial}{\partial t} + g^i\frac{\partial}{\partial x_i}\right)\right) - \mathrm{pr}^*\alpha\left(\frac{\partial f}{\partial t}\frac{\partial}{\partial t} + \frac{\partial g^i}{\partial t}\frac{\partial}{\partial x_i} + g^i\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial x_i}\right]\right) \\
&\overset{(2)}{=} \frac{\partial}{\partial t}\left(g^i\alpha\left(\frac{\partial}{\partial x_i}\right)\right) - \mathrm{pr}^*\alpha\left(\frac{\partial g^i}{\partial t}\frac{\partial}{\partial x_i}\right) \\
&\overset{(3)}{=} \frac{\partial g^i}{\partial t}\alpha\left(\frac{\partial}{\partial x_i}\right) - \frac{\partial g^i}{\partial t}\alpha\left(\frac{\partial}{\partial x_i}\right) \\
&= 0.
\end{aligned}
$$

Here, we used in (1) that $D\mathrm{pr}\left[\frac{\partial}{\partial t}\right] = 0$ and the identity $[W, hZ] = W(h) + h[W, Z]$ for any two vector fields W, Z and any smooth function h, as well as antisymmetry of the Lie bracket; in (2), again the fact that $D\mathrm{pr}\left[\frac{\partial}{\partial t}\right] = 0$ and $\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial x_i}\right] = 0$; and in (3) the fact that $\frac{\partial}{\partial t}\left(\alpha\left(\frac{\partial}{\partial x_i}\right)\right) = 0$. □

In the construction of symplectic collar neighbourhoods, we will make use of the following property of the flow of Liouville vector fields:

Lemma 1.2.6. *Suppose V is a Liouville vector field on (W^{2n}, ω). Then its flow φ_t satisfies $\varphi_t^*\omega = e^t\omega$.*

Proof. Since V is Liouville, we have for any $t \in \mathbb{R}$ that

$$
\begin{aligned}
\frac{d}{dt}\varphi_t^*\omega &= \varphi_t^*\mathcal{L}_V\omega \\
&= \varphi_t^*\omega.
\end{aligned}
$$

Suppose $x_1, \ldots, x_{2n}$ are local coordinates on W. The above equation evaluated at the coordinate vector fields $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_j}$, this becomes

$$\frac{d}{dt}\varphi_t^*\omega_{ij} = \varphi_t^*\omega_{ij},$$

which is a differential equation for ordinary functions together with the initial condition $\varphi_0^*\omega_{ij} = \omega_{ij}$. Thus the solution is

$$\varphi_t^*\omega_{ij} = e^t\omega_{ij}.$$

Since this holds for any component of ω, we obtain $\varphi_t^*\omega = e^t\omega$. □

We now have all the ingredients necessary to construct our symplectic collar neighbourhoods.

Construction 1.2.7. Let (W, ω) be a symplectic manifold and V be a Liouville vector field defined in a neighbourhood of ∂W which is outward pointing along ∂W. Set $\lambda := \imath_V \omega$. Then for some small $\epsilon > 0$, there is a symplectic collar neighbourhood embedding

$$c : (\partial W \times (-\varepsilon, 0], d(e^t \lambda)) \hookrightarrow (W, \omega).$$

Proof. To construct collar neighbourhoods for the boundary of any manifold W, one only needs to show the existence of a vector field transverse to the boundary ∂W, whereafter the collar is constructed using the flow of this vector field. Thus the fact that V is outward pointing yields a collar defined by

$$c : \partial W \times (-\varepsilon, 0] \hookrightarrow W, \qquad c(p,t) := \varphi_t(p), \qquad c(p, 0) = p \quad \forall p \in \partial W,$$

where φ_t is the flow of V. Hence as $\omega = d\imath_V \omega$, it remains to show that c pulls back λ to $e^t \lambda|_{\partial W}$. In order to do this, consider a point $(p, t_0) \in \partial W \times (-\varepsilon, 0]$ and a tangent vector U at (p, t_0), which can be represented by the tangent vector at $t = 0$ of a curve

$$(\eta(t), t_0 + tu)$$

for η a suitable curve in ∂W and $u \in \mathbb{R}$. We want to compute

$$c^* \imath_V \omega_{(p,t_0)}(U) = \imath_V \omega_{\varphi_{t_0}(p)} \left(D\varphi(p, t_0)[U] \right).$$

We first look at the derivative:

$$\begin{aligned} D\varphi(p,t_0)[(\eta'(0), u)] &= \left.\frac{d}{dt}\right|_{t=0} \varphi(\eta(t), t_0 + tu) \\ &= \left.\frac{d}{dt}\right|_{t=0} \varphi(p, t_0 + tu) + \left.\frac{d}{dt}\right|_{t=0} \varphi(\eta(t), t_0) \\ &= uV(\varphi(p,t_0)) + D\varphi_{t_0}(p)[u]. \end{aligned}$$

Thus we see that

$$\imath_V \omega_{\varphi_{t_0}(p)}(D\varphi(p,t_0)[U]) = \imath_V \omega_{\varphi_{t_0}(p)}(D\varphi_{t_0}(p)[u]) = \varphi_{t_0}^*(\imath_V \omega)(u).$$

In general for a diffeomorphism φ, we have

$$\varphi^* \imath_V \omega = \imath_{\varphi^* V} \varphi^* \omega,$$

where $\varphi^* V(y) = D\varphi^{-1}(\varphi(y))[V(\varphi(y))]$. Hence we compute

$$\begin{aligned} \varphi_t^* V(x) &= D\varphi_t^{-1}(\varphi_t(x))[\varphi_t'(x)] \\ &= \left.\frac{d}{dt'}\right|_{t'=0} \varphi_t^{-1}(\varphi_{t+t'}(x)) \\ &= \left.\frac{d}{dt'}\right|_{t'=0} \varphi_{t'}(x) \\ &= V(\varphi_0(x)) \\ &= V(x). \end{aligned}$$

Thus $\varphi_t^* V = V$, and by Lemma 1.2.6, we have $\varphi_t^* \omega = e^t \omega$, so that

$$\varphi_{t_0}^*(\imath_V \omega) = e^{t_0} \imath_V \omega.$$

Restricting to ∂W proves the claim. ☐

With minor modifications to this construction, one can construct symplectic cylindrical neighbourhoods of hypersurfaces $\Sigma \subset W$ near which there exists a transverse Liouville vector field.

1.3 Flavours of Symplectic Fillings

Having seen how symplectic manifolds with Liouville vector fields near their boundary give rise to contact manifolds, we are interested in the converse; namely, given a contact manifold, is there a symplectic manifold whose boundary is the given contact manifold, and moreover in such a way that the symplectic structure induces the contact structure?

We begin slightly more generally by discussing symplectic cobordisms.

Definition 1.3.1. Let (M_-, ξ_-) and (M_+, ξ_+) be compact oriented contact manifolds so that the contact structures $\xi_\pm$ are positive. Then, a cobordism W between M_- and M_+ is called

- a **weak symplectic cobordism** if W admits a symplectic form ω so that $\omega|_{\xi_-} < 0$ and $\omega|_{\xi_+} > 0$;
- a **strong symplectic cobordism** if there exists a Liouville vector field $V \in \mathfrak{X}(W)$ for a symplectic form ω on a neighbourhood of $\partial W = \partial M_- \sqcup M_+$ which is inward pointing on M_-, and outward pointing on M_+, so that the induced contact structure on $M_\pm$ by V agrees with $\xi_\pm$;
- an **exact symplectic cobordism**, or a **Liouville cobordism**, if it is a strong symplectic cobordism for which V is globally defined;
- a **Stein cobordism** if W admits the structure of a Stein domain; that is, a complex structure J and a J-convex Morse function $\phi : W \to \mathbb{C}$ (see Definition 1.4.6) that has M_- and M_+ as regular level sets. The Stein structure should be such that the contact structures induced on the boundary agrees with $\xi_\pm$. See Section 1.4.2, also for Weinstein cobordisms.
- a **Weinstein cobordism** if W admits a Weinstein structure: an exact symplectic form ω, together with a Liouville vector field which is gradient like for an exhausting Morse function ϕ. The function ϕ should have $M_\pm$ as regular level sets, and the induced contact structure on $M_\pm$ should agree with $\xi_\pm$.

We then call M_- the **concave** boundary, and M_+ the **convex** boundary of W.

Remark 1.3.2. The notion of weak symplectic cobordism as stated in this definition is limited to contact manifolds $M_\pm$ of dimension 3, as only then, the contact hyperplanes $\xi_\pm$ are 2-dimensional so that ω can be evaluated on them. It is not immediately obvious how to extend this notion to higher dimensions, though Massot, Niederkrüger, and Wendl have proposed in [MNW13] the notion of *weak convexity.*

Note that the notions of cobordisms above are increasingly restrictive. Strong symplectic fillings satisfy the orientation requirements of weak symplectic fillings by Remark 1.2.3. Stein cobordisms turn out to have a symplectic form $-dd^{\mathbb{C}}\phi$ compatible with J, so that the gradient of ϕ with respect to the induced metric is Liouville, and thus they are also Liouville cobordisms. Stein and Weinstein cobordisms are in fact equivalent notions as evidenced by deep theorems by Cieliebak and Eliashberg [CE10]. Some results along this line are discussed in Section 1.4.3.

The primary fillings of interest in this thesis will be Stein and strong symplectic fillings. Strong symplectic cobordisms are well-behaved in that they are transitive, by which we mean the following:

Proposition 1.3.3 ([Gei08, Proposition 5.2.5])**.** *Suppose we are given contact manifolds (M_-, ξ_-), (M, ξ) and (M_+, ξ_+) in such a way that there are strong symplectic cobordisms (W_-, ω_-) and (W_+, ω_+) from M_- to M from M to M_+, respectively.*

Then gluing W_- to W_+ along M gives a strong symplectic cobordism from (M_-, ξ_-) to (M_+, ξ_+).

Proof. Let $j_\pm : M \hookrightarrow W_\pm$ be inclusions, $V_\pm$ be corresponding Liouville vector fields near M, and define corresponding contact forms

$$\alpha_\pm := j_\pm^*(\imath_{V_\pm}\omega_\pm).$$

Since α_- and α_+ both induce the same contact structure ξ on M, one has to be a multiple of the other by a nonvanishing function, so we can find $f \in C^\infty(M)$ such that

$$\alpha_+ = e^f \alpha_-. \tag{1.1}$$

As M is compact, f attains its extrema, and thus scaling ω_+ (and thereby α_+) with a large positive constant, we may assume that $e^f > 1$, or, equivalently, that $f > 0$.

Now take two symplectic collar neighbourhoods of M in (W_+, ω_+) and (W_-, ω_-):

$$((-\varepsilon, 0] \times M), d(e^t\alpha_-)) \hookrightarrow W_-,$$

and

$$([0, \varepsilon) \times M), d(e^{t+f}\alpha_-)) \hookrightarrow W_+.$$

Note that $d(e^{t+f}\alpha_-) = d(e^t\alpha_+)$ by Equation (1.1). Set

$$W_0 = \{(t,x) \in \mathbb{R} \times M \mid 0 \leq t \leq f(x)\}$$

and endow it with the symplectic form $d(e^t\alpha_-)$. We can thus consider it as a submanifold of W_-. Identifying

$$W_- \ni (0,x) \sim (0,x) \in W_0$$

and

$$W_0 \ni (f(x),x) \sim (0,x) \in W_+$$

produces a symplectic manifold

$$W_- \cup_M W_0 \cup_M W_+$$

that serves as a cobordism, as desired. It is compact as we may view it as the image of $W_- \sqcup W_+ \sqcup W_0$ under the (continuous) quotient projection.

Let us now verify that the glued manifold inherits a global symplectic form. As the collar embeddings are symplectic, we may check well-definedness directly on the collar neighbourhoods. To check that the symplectic form is well-defined under the identifications made between W_0 and W_+, it suffices to check that

$$d(e^t\alpha_+)_{(0,x)} = d(e^t\alpha_-)_{(f(x),x)}, \quad \forall\, x \in M.$$

This, however, is true since

$$d(e^t\alpha_+)_{(0,x)} = d(e^{t+f}\alpha_-)_{(0,x)} = d(e^f\alpha_-)_x = d(e^t\alpha_-)_{(f(x),x)}.$$

The case is clear for identifications made between W_0 and W_- since there are no nontrivial identifications and the symplectic form is the same. □

The setting in the rest of this thesis will not be that of symplectic cobordisms, rather that of symplectic fillings.

Definition 1.3.4. A (weak/ strong/ exact/ Stein/ Weinstein) cobordism from the empty set to a contact manifold (M,ξ) is called a **(weak/ strong/ exact/ Stein/ Weinstein) symplectic filling**.

Example 1.3.5. Consider the unit ball

$$B^{2n+2} = \{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \leq 1\},$$

together with the restriction of the standard symplectic form

$$\omega_0 = \sum_{i=1}^{2n+2} dx_i \wedge dy_i.$$

We have seen that

$$V(\mathbf{x},\mathbf{y}) = \frac{1}{2}\sum_{i=1}^{n+1} x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}$$

is a Liouville vector field for ω_0, and it is evidently outward pointing on $S^{2n+1} = \partial B^{2n+2}$. The kernel of the associated primitive was defined as the standard contact structure on S^{2n+1}, so (B^{2n+2}, ω_0) is an exact symplectic filling of $(S^{2n+1}, \xi_{\mathrm{can}})$.

1.4 Stein and Weinstein Manifolds

Here, we would like to formally define the aforementioned notions of Stein and Weinstein fillability for contact manifolds and discuss their complex and symplectic aspects. To this end, we follow [CE10], and begin by recalling some definitions from complex geometry.

1.4.1 J-Convexity

Definition 1.4.1. Let (W, J) be an almost complex manifold. The almost complex structure J is called **tame** with respect to a symplectic form $\omega \in \Omega^2(W)$ on W if

$$\omega(v, Jv) > 0$$

for all nonzero tangent vectors $v \in TW$.

If, in addition, ω is J-invariant in the sense that

$$\omega(Ju, Jv) = \omega(u, v)$$

for all $u, v \in TW$, then we say that ω and J are **compatible**.

Remark 1.4.2. Recall that for $\omega \in \Omega^2(W)$ a symplectic form compatible with the almost complex structure J on W, one obtains a Riemannian metric on W by setting

$$g(u, v) := \omega(u, Jv).$$

Definition 1.4.3. An almost complex structure J is called **integrable** if there exists an atlas of J-holomorphic charts on W^n, that is, charts to $\mathbb{C}^n$ whose transition functions are holomorphic.

A Stein manifold will be a manifold that is equipped with a J-*convex* function. This section collects the relevant definitions.

Definition 1.4.4. Let (W, J) be an almost complex manifold and $\phi \in C^\infty(W)$ be a smooth function. We associate to ϕ the 2-form

$$\omega_\phi := -dd^{\mathbb{C}}\phi.$$

The operator $d^{\mathbb{C}}$ is defined by

$$d^{\mathbb{C}}\phi(X) := d\phi(JX)$$

for $X \in \mathfrak{X}(W)$.

Two natural questions present themselves at this point.

1. When is ω_ϕ symplectic?
2. When is $g_\phi = \omega_\phi(\cdot, J\cdot)$ a metric?

Starting with the second question, it turns out that in general, g_ϕ need not even be a symmetric tensor. It evidently is symmetric, however, if ω_ϕ is J-invariant. A sufficient condition for J-invariance is integrability.

Lemma 1.4.5. *If (W^{2n}, J) is a complex manifold (i.e. J is integrable), then ω_ϕ is J-invariant.*

Proof. On any complex manifold with complex coordinates $z_k = x_k + iy_k$, define the 1-forms

$$dz_k = dx_k + idy_k, \qquad d\overline{z_k} = x_k - idy_k.$$

Then any 1-form can be written as a $C^\infty(W)$-linear combination of these forms.

Further define the complex valued $(1,1)$-form

$$\partial\overline{\partial}\phi := \sum_{j,k=1}^{n} \frac{\partial^2 \phi}{\partial z_j \partial \overline{z_k}} dz_j \wedge d\overline{z_k}.$$

Note that

$$dz_j \circ i = idz_j, \qquad d\overline{z_j} \circ i = -id\overline{z_j},$$

and use this to compute

$$d^{\mathbb{C}}\phi = \sum_{j=1}^{n} \left(\frac{\partial}{\partial z_j}(\phi) dz_j \circ i + \frac{\partial}{\partial \overline{z_j}}(\phi) d\overline{z_j} \circ i \right) = \sum_{j=0}^{n} \left(i\frac{\partial}{\partial z_j}(\phi) dz_j - i\frac{\partial}{\partial \overline{z_j}} d\overline{z_j} \right),$$

and thus

$$-dd^{\mathbb{C}}\phi = 2i \sum_{j,k=0}^{n} \frac{\partial^2 \phi}{\partial z_j \partial z_k} dz_k \wedge d\overline{z_j} = 2i\partial\overline{\partial}\phi.$$

Since $\partial\overline{\partial}\phi$ is i-invariant, so is ω_ϕ. □

Addressing the first question, J-convexity is the concept that handles nondegeneracy of ω_ϕ.

Definition 1.4.6. A function $\phi : W \to \mathbb{R}$ on an almost complex manifold (W, J) is called **J-convex** if ω_ϕ tames J, that is, if

$$\omega_\phi(v, Jv) > 0$$

for all nonzero tangent vectors v.

The function ϕ is called **exhausting** if it is proper and bounded from below.

A J-convex function ϕ on an almost complex manifold thus gives rise to a symplectic form ω_ϕ, and moreover , if J is integrable, to a Riemannian metric g_ϕ. We also obtain that the gradient of ϕ with respect to this metric is Liouville:

Lemma 1.4.7. *Let ϕ be a J-convex function on a complex manifold (W, J) and set*

$$\omega_\phi := -dd^{\mathbb{C}}\phi, \quad \lambda_\phi := -d^{\mathbb{C}}\phi, \quad V_\phi := \mathrm{grad}_\phi.$$

Then $\omega_\phi = d\lambda_\phi$ is a symplectic form with Liouville vector field V_ϕ.

Proof. Recall that ω_ϕ is symplectic by definition of J-convexity and the assumption that J is integrable. By definition of the gradient, we have for any $Y \in \mathfrak{X}(W)$

$$d^{\mathbb{C}}\phi(Y) = d\phi(JY) = g_\phi(\mathrm{grad}_\phi, JY) = -\omega_\phi(\mathrm{grad}_\phi, Y) = -\imath_{V_\phi}\omega_\phi(Y).$$

Thus $\lambda_\phi = \imath_{V_\phi}\omega_\phi$, and $\mathcal{L}_{V_\phi}\omega_\phi = d\lambda_\phi = \omega_\phi$. $\square$

1.4.2 Liouville, Stein, and Weinstein

In this section, we will talk about Stein and Weinstein manifolds. Before we do so, we define Liouville manifolds. The notion of Liouville domains in particular is relevant later on, as their boundary is always of contact type and they always admit symplectic collars.

Definition 1.4.8. A **Liouville manifold** $(W, \omega = d\lambda, V)$ is an exact symplectic manifold (W, ω) together with a Liouville vector field V defined by $\lambda = \imath_V\omega$ such that

- the Liouville vector field V is complete;
- W is convex in the sense that there exists an exhaustion of W by compact subsets $W^k \subset W$ with smooth boundaries along which V is outward pointing, so that $W^1 \subset W^2 \subset W^3 \subset \dots$ and $W = \bigcup_{k=1}^{\infty} W^k$.

We will often suppress the Liouville vector field from the notation and denote Liouville manifolds simply by $(W, d\lambda)$.

Remark 1.4.9. If W is compact, then the second item implies that $W = W^k$ for some large k, and thus the Liouville vector field is outward pointing along ∂W.

We are now ready to formally define the cobordisms from Definition 1.3.1.

Definition 1.4.10. A **Liouville cobordism** (W, ω, V) from M_- to M_+ is a compact exact symplectic manifold $(W, \omega = d\lambda)$ with boundary $\partial W = M_- \sqcup M_+$ and a globally defined Liouville vector field V which points transversely inward on M_- and outward on M_+.

A Liouville cobordism with $M_- = \emptyset$ is called a **Liouville domain**.

Remark 1.4.11. A Liouville domain can be considered as a compact Liouville manifold. Note also that Liouville domains always admit symplectic collar neighbourhoods as the symplectic form is globally exact and V is outward pointing.

Definition 1.4.12. A **Stein manifold** (W, J, ϕ) is a complex manifold (W, J) together with an exhausting J-convex Morse function ϕ. A **Stein cobordism** (W, J, ϕ) from M_- to M_+ is a Stein manifold with $\partial W = M_- \sqcup M_+$ such that $M_\pm$ are regular level sets of ϕ. A Stein cobordism with $M_- = \emptyset$ is called a **Stein domain**.

If the manifold W is fixed, we refer to a tuple (J, ϕ) making (W, J, ϕ) into a Stein manifold as a **Stein structure** on W.

Remark 1.4.13. Stein manifolds are Liouville manifolds: the form ω_ϕ is symplectic as ϕ is J-convex, and as J is integrable, the induced metric g_ϕ is indeed a metric (see the discussion after Definition 1.4.4). The gradient vector field $V_\phi := \mathrm{grad}_\phi$ with respect to g_ϕ is Liouville by Lemma 1.4.7.

As for the convexity condition, the suggestively termed property of ϕ being exhausting implies that the sets $W^k = \phi^{-1}([-\infty, d_k])$ for $d_k \to \infty$ an increasing sequence of regular values provide a compact exhaustion of W. The vector field V_ϕ is transverse to the level sets of ϕ, which are the boundaries of the W^k, and indeed outward pointing as d_k is an increasing sequence.

Definition 1.4.14. A **Weinstein manifold** $(W, \omega = d\lambda, V, \phi)$ consists of an exact symplectic manifold $(W, \omega = d\lambda)$ with a complete Liouville vector field V which is gradient-like for an exhausting Morse function $\phi : W \to \mathbb{R}$.

A **Weinstein cobordism** (W, ω, V, ϕ) from M_- to M_+ is a Liouville cobordism (W, ω, V) such that V is gradient-like for the Morse function $\phi : W \to \mathbb{R}$ which is constant on the boundary. A Weinstein cobordism with $M_- = \emptyset$ is called a **Weinstein domain**.

A triple (ω, V, ϕ) making (W, ω, V, ϕ) into a Weinstein manifold is called a **Weinstein structure** on W.

Remark 1.4.15. The gradient-like vector field V in a Weinstein manifold $(W, \omega = d\lambda, V, \phi)$, or the gradient V_ϕ in a Stein manifold (W, J, ϕ), is always transverse to the level sets of ϕ. Thus the primitive λ of ω (or λ_ϕ of ω_ϕ, respectively) restricted to any level set induces a contact structure. In particular, the boundary of Stein and Weinstein domains carries a contact structure induced by the Stein or Weinstein structure.

Definition 1.4.16. A **Liouville/ Stein/ Weinstein filling** (W, ω) of a contact manifold (M, ξ) is a Liouville/ Stein/ Weinstein domain with $\partial W = M$ so that the induced contact structure agrees with ξ.

It is immediate that Weinstein manifolds are Liouville manifolds from the properties of V. Let us give some examples of Weinstein manifolds as found in [CE10, Example 11.12].

Example 1.4.17.

1. The canonical Weinstein structure on $\mathbb{C}^n$ is defined by

$$\omega_0 = \sum_{k=1}^{n} dx_k \wedge dy_k, \quad V_0 = \frac{1}{2}\sum_{k=1}^{n}\left(x_k\frac{\partial}{\partial x_k} + y_k\frac{\partial}{\partial y_k}\right), \quad \phi_0 = \frac{1}{4}\sum_{k=1}^{n}(x_k^2+y_k^2).$$

   The corresponding primitive of ω_0 is $\lambda_0 = \frac{1}{2}\sum_{k=1}^{n}(x_k dy_k - y_k dx_k)$. Note that the function ϕ_0 is Morse of index 0, and is bounded from below by 0. Preimages of closed intervals are closed balls in $\mathbb{C}^n$, so ϕ_0 is proper and thus exhausting. The vector field V_0 is precisely grad_{ϕ_0}.

2. The cotangent bundle T^*Q of a closed manifold Q^n carries a Weinstein structure. Suppose $q = (q_1, \ldots, q_n)$ are local coordinates on Q, and denote by (q, p) the induced coordinates on T^*Q. Then the following data define a Weinstein structure:

$$\omega_{\mathrm{can}} = d\lambda_{\mathrm{can}}, \quad V_0 = p\frac{\partial}{\partial p}, \quad \phi_0 = \frac{1}{2}|p|^2.$$

   Here, $\lambda_{\mathrm{can}} = pdq$ denotes the canonical Liouville form on T^*Q. The vector field V_0 is easily verified to be Liouville, and moreover, it is precisely the gradient of ϕ_0: note that the Christoffel symbols of the metric $dp \otimes dp$ on T^*Q vanish, and thus

$$\frac{\partial}{\partial p_i}|p|^2 = 2g(\nabla_{\frac{\partial}{\partial p_i}}(p_j\frac{\partial}{\partial p_j}), p_j\frac{\partial}{\partial p_j}) = 2p_i.$$

   Hence $\mathrm{grad}_{\phi_0} = p\frac{\partial}{\partial p} = V_0$.

   Note, however, that ϕ_0 is not Morse, but rather Morse-Bott. One could in fact relax the definition to allow for Morse-Bott functions, but we will slightly perturb ϕ_0 in order to obtain a Weinstein structure in the sense of Definition 1.4.14. To this end, consider any Riemannian metric on Q and a Morse function $f : Q \to \mathbb{R}$. The Hamiltonian vector field X_F of $F(q,p) := p(\mathrm{grad}_f(q))$ coincides with grad_f along the zero section of T^*Q: in coordinates,

$$F(q,p) = p_i\frac{\partial}{\partial q_i}(f), \quad dF = \frac{\partial}{\partial q_i}(f)dp_i + p_i\frac{\partial^2 f}{\partial q_i \partial q_j}dq_j.$$

   Thus the Hamiltonian vector field is

$$X_F = -p_i\frac{\partial^2 f}{\partial q_i \partial q_j}\frac{\partial}{\partial p_i} + \frac{\partial f}{\partial q_i}\frac{\partial}{\partial q_i},$$

   where the last summand is recognized as grad_f in local coordinates. Hence $V := p\frac{\partial}{\partial p} + X_F$ is Liouville and gradient-like for the Morse function $\phi(q,p) := \frac{1}{2}|p|^2 + f(q)$ for f small enough.

Remark 1.4.18. For Weinstein manifolds, one can describe a symplectic handlebody decomposition of the underlying manifold in the following sense: attaching a so-called **Weinstein handle** to a Weinstein manifold yields a new manifold so that the symplectic form, Liouville vector field and exhausting Morse function extend over the handle in such a way that the new manifold is still a Weinstein manifold. The Morse function picks up precisely one critical point in the handle, corresponding to a zero of the Liouville vector field. Moreover, the manifold obtained this way is unique up to an appropriate notion of homotopy. See [Wei91] for the original construction.

It turns out that Weinstein domains are subject to a strong topological constraint.

Lemma 1.4.19 ([CE10, Lemma 2.21])**.** *Let $(W^{2n}, \omega, V, \phi)$ be a Weinstein domain. Then the index of each critical point of ϕ is at most n.*

Proof. Denote the flow of V by φ_t and recall that since V is Liouville, we have $\varphi_t^*\omega = e^t\omega$, so we may write

$$\omega = e^{-t}\varphi_t^*\omega.$$

Suppose p is a critical point of ϕ and $W^s(p)$ is the stable manifold associated to p. For any $q \in W^s(p)$, we hence have $\varphi_t(q) \to p$ for $t \to \infty$. As in particular $\omega_q = e^{-t}\varphi_t^*\omega_q$ for all t, we obtain

$$\omega_q = \lim_{t\to\infty} e^{-t}\varphi_t^*\omega_q = 0 \cdot \omega_p = 0.$$

Thus ω vanishes on $W^s(p)$. This implies that $W^s(p)$ is an isotropic submanifold of W. Hence the dimension of $W^s(p)$, which equals to $2n - \mathrm{ind}(\phi)$ because V is gradient-like for ϕ, is at most n, as desired. □

Thus, any Weinstein domain W^{2n} admits a handlebody decomposition with handles of index no greater than n.

1.4.3 From Stein to Weinstein and Back: A Brief Stopover

In their book with the same title [CE10], Cieliebak and Eliashberg explain how Stein and Weinstein structures are equivalent up to deformation. We refer to the reader to the book for the proof. In what follows, we will outline the main results.

The Road from Stein to Weinstein

To a Stein cobordism (W, J, ϕ), we can always associate a Weinstein cobordism structure. Denote this by the functor $\mathfrak{W}$:

$$\mathfrak{W}(J, \phi) := (\omega_\phi, V_\phi, \phi),$$

where $\omega_\phi = -dd^{\mathbb{C}}\phi$, and $V_\phi = \mathrm{grad}_\phi$ with respect to the metric g_ϕ induced by ω_ϕ. This is indeed a Weinstein manifold: ω_ϕ is symplectic by J-convexity of ϕ

and J-compatible as J is integrable. We have seen in Lemma 1.4.7 that V_ϕ is a Liouville vector field. The gradient itself is evidently gradient-like for ϕ, and ϕ is constant on ∂W already by the definition of a Stein cobordism.

Remark 1.4.20. The considerations regarding the indices of handle decompositions of a Weinstein domain hence carry over to Stein domains. Therefore, a necessary condition for a manifold W^{2n} to admit a Stein structure is that there is a handlebody decomposition of W where no handle has index $\geq n$.

The Road from Weinstein to Stein

Given a Weinstein structure (ω, V, ϕ) on a manifold W, it is highly nontrivial to construct a complex structure J on W making (W, J, ϕ) into a Stein manifold, and in fact, this is one of the main results of [CE10].

Theorem 1.4.21 ([CE10, Theorem 1.1(a), Theorem 13.9])**.** *Let (W, ω, V, ϕ) be a Weinstein manifold. Then there exists a Stein structure (J, ϕ) on W so that the Weinstein structures*

$$(\omega, V, \phi) \text{ and } \mathfrak{W}(J, \phi)$$

are homotopic. Note that the function ϕ is fixed.

As a consequence of this theorem, the notions of Stein and Weinstein fillability are seen to be equivalent. Indeed, given a Weinstein filling (W, J, ϕ) of a contact manifold, we may apply Theorem 1.4.21 to deform it into a Stein domain while keeping the contact structure induced by the contact form $-d^{\mathbb{C}}\phi$ invariant.

Similarly, the fact that any Stein filling is also a Weinstein filling follows from the observation that any Stein domain (W, J, ϕ) is a Weinstein domain with Weinstein structure $\mathfrak{W}(J, \phi)$, where the J-convex function ϕ, and thereby the contact structure on the boundary, remains unchanged.

Another important theorem for us concerns the existence of Stein (and hence, Weinstein) structures.

Theorem 1.4.22 ([CE10, Theorem 8.15])**.** *Let W^{2n} be an open smooth manifold of dimension $2n \neq 4$ which admits an almost complex structure J and an exhausting Morse function ϕ with no critical points of index greater than n. Then J is homotopic through almost complex structures to an integrable almost complex structure $\tilde{J}$ such that ϕ can be reparametrized to be $\tilde{J}$-convex. That is, W admits a Stein structure.*

A topological analogue in dimension 4 is due to Gompf [Gom98]:

Theorem 1.4.23 ([Gom98], [CE10, Theorem 1.6])**.** *Let V be an oriented open topological 4-manifold which admits a (possibly infinite) handlebody decomposition without handles of index greater than 2. Then V is homeomorphic to a Stein surface.*

The significance of these theorems is that they allow us to see that the total space of a Lefschetz fibration, introduced in the following chapter, admits the structure of a Stein domain.

Chapter 2

Lefschetz and Lefschetz-Bott Fibrations

Lefschetz fibrations and their generalizations, Lefschetz-Bott fibrations, will be the main tool in this text to obtain symplectic fillings, although we will not see this until Section 3.3.

Lefschetz fibrations can be considered as a complex analogue of Morse functions, and can be used to give a topological description of the total space in the same fashion, as will be explained in Section 2.1. In the second part, we will review Lefschetz fibrations in the context of symplectic manifolds, before generalizing to symplectic Lefschetz-Bott fibrations in Section 2.3.

2.1 Topological Lefschetz Fibrations

Definition 2.1.1. Let E^{2n} be an even-dimensional manifold and S be a compact surface. A proper map $\pi : E \to S$ is called a **Lefschetz fibration** if

- all its critical points E^{crit} are contained in $\mathrm{int}(E)$;
- near each critical point in E and each critical value in S, there exist charts (U, σ) and (V, τ), respectively, in which

$$\begin{aligned} \tau \circ \pi \circ \sigma^{-1} : \mathbb{C}^n &\to \mathbb{C} \\ (z_1, \dots, z_n) &\mapsto z_1^2 + \dots + z_n^2. \end{aligned}$$

Let us call such coordinates **Lefschetz charts**.

One sometimes calls a Lefschetz fibration **positive** if the complex charts are orientation preserving, which we shall assume henceforth.

We will at times simply write (E, π) for Lefschetz fibrations whose base is clear from the context, and refer to Lefschetz fibrations as **topological Lefschetz fibrations** in contrast to *symplectic Lefschetz fibrations* introduced in Definition 2.2.1

Remark 2.1.2. The second item in Definition 2.1.1 implies that critical points of Lefschetz fibrations are isolated, i.e, E^{crit} is a finite set of points.

Let us start right away with an example.

Example 2.1.3. Consider the polynomial $f_k \in \mathbb{C}[z_1, \ldots, z_{n+1}]$ defined by

$$f_k(z_1, \ldots, z_{n+1}) = z_1^2 + \ldots + z_n^2 + z_{n+1}^{k+1}$$

for some integer $k \geq 1$, and let

$$V_k(\epsilon) = \{\mathbf{z} \in \mathbb{C}^{n+1} \mid f_k(\mathbf{z}) = \epsilon\}$$

for some $\epsilon > 0$.

Claim 2.1.4. *The projection map*

$$\begin{aligned} \pi : V_k(\epsilon) &\to \mathbb{C} \\ (z_1, \ldots, z_{n+1}) &\mapsto z_{n+1} \end{aligned}$$

is a Lefschetz fibration.

Proof. Let us first find the critical points of π. Note that

$$z_{n+1}^{k+1} = \epsilon - z_1^2 - \ldots - z_n^2.$$

Differentiating this equation, it follows that

$$(k+1) z_{n+1}^k dz_{n+1} = -2(z_0 dz_0 + \ldots + z_n dz_n).$$

Suppose now that $\mathbf{z} = (z_1, \ldots, z_{n+1}) \in V_k(\epsilon)$ is a critical point of π, which means $d\pi_{\mathbf{z}} = (dz_{n+1})_{\mathbf{z}} = 0$.

Case I: $z_{n+1} = 0$

If $z_{n+1} = 0$, then also $z_i = 0$ for $i \leq n$, but $\mathbf{0} = (0, \ldots, 0) \notin V_k(\epsilon)$. Therefore, critical points have nonvanishing z_{n+1}-coordinate.

Case II: $z_{n+1} \neq 0$

If $z_{n+1} \neq 0$, this implies that $dz_{n+1} = d\pi$ vanishes for all $(0, \ldots, 0, z_{n+1}) \in V_k(\epsilon)$. Hence for μ_{k+1} a $(k+1)$-th root of unity, the points

$$\lambda_l := (0, \ldots, 0, \mu_{k+1}^l), \quad l = 0, \ldots, k,$$

are the only critical points of π.

It remains to find complex charts to bring π into the standard form. Define

$$\rho_{\mathbf{z}} = |\epsilon - z_1^2 - \ldots - z_n^2|, \qquad \varphi_{\mathbf{z}} = \arg(\epsilon - z_1^2 - \ldots - z_n^2).$$

We can parametrise $V_k(\epsilon)$ near λ_l by

$$\begin{aligned} \sigma_l : \mathbb{C}^n &\longrightarrow V_k(\epsilon) \\ (z_1, \ldots, z_n) &\longmapsto \left(z_1, \ldots z_n, \rho_{\mathbf{z}}^{\frac{1}{k+1}} \exp\left(i \frac{\varphi_{\mathbf{z}} + 2\pi l}{k+1} \right) \right), \end{aligned}$$

which is a complex chart when restricted to a small enough neighbourhood of $\mathbf{0} \in \mathbb{C}^n$. Note that $\sigma_l(\mathbf{0}) = \lambda_l$. Then

$$\pi \circ \sigma_l(\mathbf{z}) = \rho_{\mathbf{z}}^{\frac{1}{k+1}} \exp\left(i \frac{\varphi_{\mathbf{z}} + 2\pi l}{k+1} \right),$$

and $\pi \circ \sigma_l$ maps a neighbourhood of zero in $\mathbb{C}^n$ to a neighbourhood of $\mu_l \in \mathbb{C}$. On a small neighbourhood of μ_l, the map

$$\begin{aligned} \tau : \mathbb{C} &\to \mathbb{C} \\ w &\mapsto 1 - w^{k+1} \end{aligned}$$

is biholomorphic. Since

$$\tau \circ \pi \circ \sigma_l(\mathbf{z}) = z_1^2 + \ldots + z_n^2,$$

the maps σ_l and τ provide the desired chart description near each critical point λ_l, and thus π is a Lefschetz fibration. $\square$

2.1.1 Vanishing Cycles

We now study the fibers of Lefschetz fibrations. Recall the following fundamental lemma:

Lemma 2.1.5 (Ehresmann fibration lemma)**.** *Let $\pi : X \to Y$ be a proper submersion between connected smooth manifolds. Then π is a fiber bundle, that is, locally trivial. In particular, the fibers are all diffeomorphic.*

Away from the critical points, a Lefschetz fibration $\pi : E \to S$ is, by definition, a smooth submersion, and hence the fibers E_z for $z \in S \setminus \pi(E^{\text{crit}})$ are diffeomorphic. Such fibers are called **regular**, as opposed to **critical** or **singular fibers** E_{x_0} for $x_0 \in \pi(E^{\text{crit}})$. We will denote the abstract regular fiber of a Lefschetz fibration (E, π) by the letter F.

The critical fibers can be better understood through vanishing cycles.

Definition 2.1.6. Let $p \in E^{\text{crit}}$ be a critical point and (U, σ), (V, τ) be Lefschetz charts near p and $\pi(p)$, respectively. On U, the regular fibers are then diffeomorphic to

$$(\tau \circ \pi \circ \sigma^{-1})^{-1}(z)$$

for some $z \in V \setminus \{0\}$. Multiplying τ by some unit complex number, we may assume that $z = t > 0$ lies in $\mathbb{R}$, and by possibly scaling (U, σ), we may assume σ to be a diffeomorphism between U and the closed unit disk $\mathbb{D}^{2n+2} \subset \mathbb{C}^{n+1}$. The fiber near U will then be diffeomorphic to

$$\begin{aligned} E_t \cap U &\cong \{\mathbf{z} \in \mathbb{C}^n \mid z_1^2 + \ldots + z_n^2 = t\} \cap \mathbb{D}^{2n+2} \\ &= \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 = t,\ \langle \mathbf{x}, \mathbf{y} \rangle = 0,\ \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \le 1\}, \end{aligned}$$

where we identify $\mathbf{z} = \mathbf{x} + i\mathbf{y}$. Define the **vanishing cycle** γ of the critical point p to be the real part of this set, which is

$$\gamma = \{(\mathbf{x}, \mathbf{0}) \mid \|\mathbf{x}\|^2 = t\} \cong S^{n-1}.$$

Note that for $n = 2$, $E_t \cap U$ is diffeomorphic to a one-sheeted hyperboloid; in this case, the vanishing cycle corresponds to the closed curve around its "waist". From this definition, we see how to obtain the singular fibers from nearby regular fibers:

Proposition 2.1.7 ([ÖS04b, Section 10.1])**.** *Let $\pi : E \to S$ be a Lefschetz fibration with regular fiber F. If p is a critical point of π, the singular fiber over $\pi(p)$ is obtained by considering nearby fibers E_t for $t > 0$ and taking $t \to 0$, or equivalently, by collapsing the vanishing cycle $\gamma \subset F$ of p.*

Let us describe the fibers near p more generally. Scaling $\mathbf{x}$ by setting

$$\mathbf{x}' = \frac{1}{\sqrt{t + \|\mathbf{y}\|^2}} \cdot \mathbf{x},$$

we obtain that

$$\begin{aligned} F_t \cap U &\cong \left\{ (\mathbf{x}', \mathbf{y}) \mid \|\mathbf{x}'\| = 1,\ \langle \mathbf{x}', \mathbf{y} \rangle = 0,\ \|\mathbf{y}\|^2 \le \frac{1-t}{2} \right\} \\ &= D_r T^* S^{n-1}, \end{aligned}$$

for $r = \frac{1-t}{2}$. The vanishing cycle thus corresponds to the zero section of $T^* S^{n-1}$.

2.1.2 The Topology of the Total Space

This section explores the analogy of Lefschetz fibrations to Morse functions by using them to give a topological description of their total space.

For this purpose, suppose $\pi : E^{2n} \to S$ is a Lefschetz fibration with regular fiber F. Consider the function

$$\pi_{\mathbb{R}} = -\mathrm{Re}(\pi).$$

On a Lefschetz chart near a critical point p, $\pi_{\mathbb{R}}$ takes the form

$$\pi_{\mathbb{R}}(\mathbf{x}, \mathbf{y}) = -x_1^2 - \ldots - x_n^2 + y_1^2 + \ldots + y_n^2.$$

Thus Lefschetz charts are Morse charts for $\pi_{\mathbb{R}}$, and each critical point of $\pi_{\mathbb{R}}$ is of index n.

If $\mathbb{D} \subset \Sigma$ is a disk containing no critical values, then $\pi^{-1}(\mathbb{D}) \cong F \times \mathbb{D}$. It is globally trivial since $\mathbb{D}$ is contractible. As we enlarge $\mathbb{D}'$ to contain a single critical value $s = \pi(p)$ and apply an isotopy such that s lies on the real axis, standard Morse theory tells us that $\pi^{-1}(\mathbb{D}')$ is diffeomorphic to $\pi^{-1}(\mathbb{D})$ with an n-handle attached to the unstable manifold of p at a subcritical level $s - t$ of $\pi_{\mathbb{R}}$. More precisely, following the Morse charts from above, this means we glue an n-handle to

$$W^u(p) \cap \pi_{\mathbb{R}}^{-1}(s-t) \cong \{(\mathbf{x}, \mathbf{0}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \|\mathbf{x}\|^2 = t\}.$$

This, however, is precisely the vanishing cycle.

Remark 2.1.8. Attaching an n-handle requires two pieces of data in order to be completely specified: an (isotopy class of an) embedding $s_0 : S^{n-1} \hookrightarrow \partial M$ corresponding to the attaching circle, and a framing of $s_0(S^{n-1})$. The above proposition has specified only the isotopy class of the embedding circle as the vanishing cycle. For more details regarding the framing, see [ÖS04b, Chapter 10].

Suppose now that $\pi : E^{2n} \to \mathbb{D}$ is a Lefschetz fibration over the disk with regular fiber F and critical values $(x_1, \ldots, x_k)$. The previous discussion proves

Proposition 2.1.9. *The total space E^{2n} admits a handlebody decomposition as*

$$E = \mathbb{D} \times F \cup (\bigcup_i H_i),$$

where each H_i is an n-handle glued to the vanishing cycle of the critical point x_i.

If E is compact, the function $\pi_{\mathbb{R}}$ is automatically an exhausting Morse function. When E has dimension 4, Theorem 1.4.23 yields the existence of a Stein structure on E. In order to use Theorem 1.4.22 in higher dimensions, E is required to admit an almost complex structure J, which cannot be assumed in general. We will return to this case in Section 3.3, see in particular Remark 3.3.3.

2.1.3 Monodromy

Monodromy is a fundamental notion describing the behaviour of a holomorphic function near a critical value. The study of singularities of holomorphic functions extends significantly beyond the scope of what will be relevant for our considerations, and we refer to [AGV88] for a more thorough treatment. The following material is adapted from Chapter 1 of the same reference.

Suppose $\pi : E \to \mathbb{D}$ is a Lefschetz fibration over the disk and choose an Ehresmann connection on E. Recall how parallel transport is defined.

If $\gamma : [a, b] \to \mathbb{D}$ is any path in the base, we may consider for any $p \in E_{\gamma(t)}$ the unique horizontal lift $X_\gamma(p) \in T_pE$ of the vector $\gamma'(t) \in T_{\gamma(t)}\mathbb{D}$ such that $D\pi(p)[X_\gamma(p)] = \gamma'(t)$. The vector field X_γ hence defines a horizontal vector field on the total space of γ^*E, and its flow φ_t defines the **parallel transport maps**

$$\begin{aligned} \rho_\gamma : E_{\gamma(a)} &\to E_{\gamma(b)} \\ \rho_\gamma(p) &= \varphi_{b-a}(p). \end{aligned}$$

Note that if π is proper, then parallel transport exists for all time.

Now fix a regular value $z_0 \in \mathbb{D}$ and let $\gamma : [0, 1] \to \mathbb{D}$ be a loop based at z_0 whose image is contained in $\mathbb{D} \setminus \pi(E^{\mathrm{crit}})$.

Parallel transport hence yields a family of maps

$$\Gamma_t := \rho_{\gamma|_{[0,t]}} : E_{z_0} \to E_{\gamma(t)}.$$

Definition 2.1.10. The map

$$\mu_\gamma := \rho_\gamma = \Gamma_1 : E_{z_0} \to E_{z_0}$$

is called the **monodromy** of the fibration π.

The monodromy is well-defined up to isotopy under different choices of representatives of homotopy classes of γ, as well as connections. To see this, suppose first that δ is a loop based at z_0 homotopic to γ via $H : [0, 1] \times [0, 1] \to \mathbb{D}$. Then $(\rho_{H(s,\cdot)})_{s\in[0,1]}$ is an isotopy between ρ_γ and ρ_δ.

As for the choice of connection, let $\tilde{\rho}$ be the parallel transport system arising through the choice of another connection and set $K_t := \tilde{\rho}_{\gamma|_{[0,t]}}$. Evidently, $\mathrm{id}_{E_{z_0}} = K_0 = \Gamma_0$. Then $K_s^{-1} \circ \Gamma_s$ is an isotopy from the identity to $K_1^{-1} \circ \Gamma_1$, whence K_1 is isotopic to Γ_1.

Remark 2.1.11. The map Γ induces a trivialization of the pullback bundle $\gamma^*E \to [0, 1]$ by

$$\begin{aligned} \Gamma : E_{z_0} \times [0, 1] &\to \gamma^*E \\ (p, t) &\mapsto \Gamma_t(p). \end{aligned}$$

From this description, it is evident that if we consider the loop γ as having domain S^1, then the pullback bundle $\gamma^*E \to S^1$ is diffeomorphic to a mapping torus whose gluing is the monodromy μ_γ:

$$\gamma^*E \cong E_{z_0} \times [0, 1]/(p, 1) \sim (\mu_\gamma(p), 0).$$

If γ bounds a disk embedded in $\mathbb{D} \setminus \pi(E^{\mathrm{crit}})$, then γ is contractible, and hence μ_γ is isotopic to the identity. However, if γ encircles a critical value in $\mathbb{D}$, the monodromy will in general not be trivial. In many cases, it will be given by what is known as a *Dehn twist*, which we describe in the next section.

2.1.4 Dehn Twists

Dehn twists describe the diffeomorphism obtained by cutting along an embedded sphere S^n, twisting by one full rotation, and regluing. In the symplectic setting, it is well known that Dehn twists generate the symplectic mapping class group of surfaces [Waj99]. It was first noticed by Arnol'd in [Arn95] that Dehn twists are symplectomorphisms with respect to the canonical symplectic structure on T^*S^n. Moreover, Seidel [Sei97] established that in certain cases, Dehn twists present a nontrivial element of the symplectic mapping class group: symplectic Dehn twists are smoothly isotopic to the identity, but not necessarily via an isotopy of symplectomorphisms.

Here we follow [Oba18] in the definition of Dehn twists adapted to the symplectic setting, starting by defining them on T^*S^n.

We will use the identification

$$T^*S^n = \{(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|\mathbf{q}\| = 1, \langle \mathbf{q}, \mathbf{p} \rangle = 0\}.$$

In these coordinates, the canonical Liouville form on T^*S^n may be written as $\lambda_0 = \sum_{i=1}^{n} p_i dq_i$, and the zero section corresponds to

$$i_0(S^n) = \{(\mathbf{q}, 0) \in T^*S^n\}.$$

Note that $i_0(S^n)$ is Lagrangian. We will now construct a Hamiltonian action on T^*S^n, through which we then define the Dehn twist. Define the Hamiltonian function

$$\begin{aligned} \mu : T^*S^n \setminus i_0(S^n) &\to \mathbb{R} \\ (\mathbf{q}, \mathbf{p}) &\mapsto \|\mathbf{p}\| . \end{aligned}$$

The Hamiltonian vector field X_μ is then

$$X_\mu := -\frac{1}{\|\mathbf{p}\|} \sum_{j=1}^{n+1} p_j \frac{\partial}{\partial q_j} - \|\mathbf{p}\| \sum_{j=1}^{n+1} q_j \frac{\partial}{\partial p_j}.$$

Claim 2.1.12. *The Hamiltonian vector field X_μ has periodic orbits.*

Proof. Letting $\mathrm{pr} : T^*S^n \to S^n$ denote the projection and $\varphi_t(q, p) = (q(t), p(t))$ be the flow of X_μ, consider $\delta(t) = \mathrm{pr} \circ \varphi_t(q, p)$. We show this is a geodesic of S^n.

Note first that

$$\begin{aligned} \langle \delta'(t), \delta'(t) \rangle &= \|D\mathrm{pr}((\mathbf{q}(t), \mathbf{p}(t)))[X_\mu((\mathbf{q}(t), \mathbf{p}(t)))]\|^2 \\ &= \left\| \|\mathbf{p}(t)\|^{-1} \sum_{j=1}^{n+1} p_j(t) \frac{\partial}{\partial q_j} \right\|^2 \equiv 1. \end{aligned}$$

Thus we have

$$\begin{aligned} 0 &= \frac{d}{dt}\left\langle \delta'(t), \delta'(t) \right\rangle \\ &= 2\left\langle \nabla_{\frac{\partial}{\partial t}}(\delta')(t), \delta'(t) \right\rangle, \end{aligned}$$

which implies $\nabla_{\frac{\partial}{\partial t}}(\delta') = 0$, so δ is a geodesic of S^n. The geodesics of S^n, however, consist of 2π-periodic great circles. □

Thus there is a Hamiltonian S^1-action on $T^*S^n \setminus i_0(S^n)$ given by $e^{it} \cdot (\mathbf{q}, \mathbf{p}) = \varphi_t(\mathbf{q}, \mathbf{p})$. From the description of the geodesics of S^n as great circles, we obtain that the action is explicitly given by

$$e^{it} \cdot (\mathbf{q}, \mathbf{p}) := (\cos(t)\mathbf{q} + \|\mathbf{p}\|^{-1} \sin(t)\mathbf{p}, -\|\mathbf{p}\| \sin(t)\mathbf{q} + \cos(t)\mathbf{p}).$$

Denote the action by

$$\begin{aligned} \sigma : S^1 &\to \mathrm{Diff}(T^*S^n \setminus i_0(S^n)) \\ e^{it} &\mapsto \sigma_t, \end{aligned}$$

and note that $\sigma_\pi(\mathbf{q}, \mathbf{p}) = (-\mathbf{q}, -\mathbf{p})$. This extends to all of T^*S^n, restricting to the antipodal map on $i_0(S^n)$, which we denote by A.

We now get to defining the prototype of a Dehn twist. Take a smooth function $\eta : \mathbb{R} \to \mathbb{R}$ such that

- $\eta(t) = 0$ for $t > t_0$ for some t_0,
- $\eta(t) + \eta(-t) = 2\pi$ for all t.

Note that this implies $\eta(0) = \pi$. The **model right-handed Dehn twist** $\tau : T^*S^n \to T^*S^n$ is now defined by

$$\tau(\mathbf{q}, \mathbf{p}) := \begin{cases} \sigma_{\eta(\|\mathbf{p}\|)}(\mathbf{q}, \mathbf{p}), & \mathbf{p} \neq 0, \\ A(\mathbf{q}, \mathbf{p}), & \mathbf{p} = 0. \end{cases}$$

Remark 2.1.13. The model right-handed Dehn twist has compact support by definition, and is a symplectomorphism of $(T^*S^n, d\lambda_{\mathrm{can}})$ by [Sei99, Section 6].

Visualising T^*S^1 as a cylinder, the action of a model Dehn twist is precisely cutting along the zero section, fully twisting one end counterclockwise, and regluing.

We proceed to define Dehn twists along any Lagrangian sphere in a symplectic manifold (W, ω), which we define as a Lagrangian submanifold L of W together with an associated **framing**, that is, a diffeomorphism

$$f : S^n \to L$$

defined up to reparameterization by orthogonal transformations. Recall the Weinstein tubular neighbourhood theorem, originally from [Wei71], as seen in [Can06]:

Theorem 2.1.14 ([Wei71], [Can06, Theorem 9.3]). *Let (W^{2n}, ω) be a symplectic manifold and $i : L \hookrightarrow W$ be a Lagrangian embedding of an n-manifold L, that is, an embedding such that $i^*\omega = 0$. Denote the zero section by $i_0 : L \to T^*L$. Then there exists $\epsilon > 0$ and a symplectic embedding $j : D_\epsilon T^*L \hookrightarrow W$ such that the following diagram commutes:*

$$\begin{tikzcd} D_\epsilon T^*L \arrow[rr, hook, "j"] & & W \\ & L \arrow[ul, hook, "i_0"] \arrow[ur, hook', "i"'] & \end{tikzcd}$$

By the Weinstein tubular neighbourhood theorem, given a Lagrangian sphere and its framing f, there is a symplectic embedding $j : D_\epsilon T^*L \hookrightarrow W$ such that $j \circ i_0 = f$. Now let $\eta : \mathbb{R} \to \mathbb{R}$ be a function as above with $t_0 = \epsilon/2$, denote by τ the corresponding model Dehn twist, and set

$$\tau_L(x) := \begin{cases} j \circ \tau \circ j^{-1}, & x \in \mathrm{Im}(j), \\ x, & x \notin \mathrm{Im}(j). \end{cases}$$

The map $\tau_L : (W, \omega) \to (W, \omega)$ is hence a compactly supported symplectomorphism and is called the **right-handed Dehn twist along** L.

With this in hand, one can prove the following for Lefschetz fibrations whose total space is 4-dimensional:

Theorem 2.1.15 ([ÖS04b, Proposition 10.1.5], [GS99, Section 8.2]). *The monodromy along a loop encircling a single critical value of a Lefschetz fibration is given by a right-handed Dehn twist along its vanishing cycle.*

See also the introduction of [AGV88] for a more explicit computation.

For Lefschetz fibrations in higher dimensions, Theorem 2.1.15 holds verbatim if one can make it into a *symplectic* Lefschetz fibration (see Definition 2.2.1 and Theorem 2.2.9). We note that the Lefschetz fibration π from Example 2.1.3 can be made symplectic with either the standard symplectic structure or a symplectic structure induced by the Fubini-Study form (see [KK16] or [Oba20]), so that by computing its vanishing cycles, we obtain the monodromy as the composition of Dehn twists along the vanishing cycles.

Example 2.1.16. Recall that we set

$$V_k(\epsilon) = \{(z_1, \ldots, z_n, z_{n+1}) \in \mathbb{C}^{n+1} \mid z_1^2 + \ldots + z_n^2 + z_{n+1}^{k+1} = \epsilon\},$$

and $\pi(z_1, \ldots, z_{n+1}) = z_{n+1}$. Fixing a critical point $\lambda_l \in V_k(\epsilon)$, the vanishing cycles are $\sigma_l(\gamma_t)$ for some $t > 0$ small enough, where

$$\gamma_t = \{\mathbf{x} \in \mathbb{C}^n \mid \mathrm{Im}(\mathbf{x}) = 0, \; \|\mathbf{x}\|^2 = t\}.$$

For $t < 1$, we have $\rho_{\mathbf{x}} = 1 - t^2 > 0$, and $\varphi_{\mathbf{z}} = 0$, so that

$$\sigma_l(\gamma_t) = \{(\mathbf{x}, (1 - t^2)) \in \mathbb{C}^{n+1} \mid \mathrm{Im}(\mathbf{x}) = 0, \; \|\mathbf{x}\|^2 = t\} =: \gamma.$$

The vanishing cycle γ is evidently independent of l, and thus all critical points λ_l have γ as their associated vanishing cycle. Consequently, by Theorem 2.2.9, the monodromy of π along $\partial\mathbb{D}$ is isotopic to τ_γ^{k+1}, the composition of $(k+1)$ right-handed Dehn twists along γ.

Note that so far, the total space of this Lefschetz fibration was not compact. To tie this together with the setting in the rest of this thesis, restrict π to the compact subdomain

$$V_k(\epsilon) \cap \mathbb{D}^{2n+2}_{1+\eta}.$$

For a regular fiber over some $t > 0$, we obtain

$$\begin{aligned}
\pi^{-1}(t) &= \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \mid z_1^2 + \ldots + z_n^2 + t^{k+1} = \epsilon, \sum_{i=1}^{n} |z_i|^2 \leq 1 + \eta - t^2\} \\
&\cong \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 = \epsilon - t^{k+1}, \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \leq 1 + \eta - t^2\} \\
&\cong \{(\mathbf{x}', \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \|\mathbf{x}'\| = 1, \|\mathbf{y}\|^2 \leq r_t\} \\
&\cong \mathbb{D}T^*S^{n-1},
\end{aligned}$$

where

$$\mathbf{x}' = \frac{\mathbf{x}}{\sqrt{\epsilon - t^{k+1} + \|\mathbf{y}\|^2}}, \text{ and } r_t = \frac{1 + \eta - t^2 - \epsilon + t^{k+1}}{2},$$

which is defined for t small enough. The vanishing cycle is hence

$$\begin{aligned}
\sigma_l(\gamma_t) &= \sigma_l(\{(\mathbf{x}, \mathbf{0}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \|\mathbf{x}\|^2 = t\}) \\
&= \{((x_1, \ldots, x_n, 1 - t), \mathbf{0}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|\mathbf{x}\|^2 = t\},
\end{aligned}$$

which is just the zero section of $\mathbb{D}T^*S^{n-1}$. Thus in conclusion, the regular fibers of π can be identified with $\mathbb{D}T^*S^{n-1}$, and the monodromy of π consists of $k+1$ right-handed Dehn twists along the zero section.

We will return to the fiber $\pi^{-1}(t)$ later on in Section 5.5, where we construct distinct fillings of the A_k-type singularity.

2.2 Symplectic Lefschetz Fibrations

Now that we are familiar with the topological properties of Lefschetz fibrations, we consider them in the context of symplectic geometry. It turns out that some more subtlety is required in their definition. The theory of symplectic Lefschetz fibrations, which is also known as *symplectic Picard-Lefschetz theory*, is essentially due to Seidel. The main theory was largely developed in [Sei03], and a comprehensive overview is contained in [Sei08], which is the main reference for this section.

Definition 2.2.1. A **symplectic Lefschetz fibration** is a tuple (E, π, Ω, J, j) consisting of

- an even dimensional manifold E;
- a smooth proper map $\pi : E \to \mathbb{C}$ whose critical points E^{crit} lie in $\text{int}(E)$;
- a closed 2-form Ω on E;
- an almost-complex structure J defined on a neighbourhood of the critical points $E^{\text{crit}} = \{q_1, \ldots, q_k\}$ of π;
- a complex structure j on a neighbourhood of the critical values in $\mathbb{C}$ compatible compatible with the standard orientation.

These are subject to the following conditions:

(i) π is (J, j)-holomorphic near E^{crit} wherever J and j are defined;

(ii) Ω is nondegenerate on the vertical bundle $T^v E = \ker D\pi$ and J-Kähler near each q_i where J is defined;

(iii) The complex Hessian at any critical point is nondegenerate as a complex quadratic form.

We will be specifically interested in symplectic Lefschetz fibrations over the unit disk $\mathbb{D} \subset \mathbb{C}$ whose fibers have nonempty boundary. In this case, we require two more conditions:

(iv) the boundary ∂E consists of the **vertical boundary** $\partial_v E$ and the **horizontal boundary** $\partial_h E$, which meet in a codimension two corner. The two boundary components are defined as

$$\partial_v E := \pi^{-1}(\partial \mathbb{D}), \text{ and } \partial_h E := \bigcup_{y \in \mathbb{D}} \partial \left(\pi^{-1}(y)\right).$$

For all $x \in \partial_h E$, we require $(\ker(D\pi(x)))^{\Omega} \subset T_x \partial_h E$.

(v) $\pi|_{\partial_v E}$ maps $\partial_v E$ submersively onto $\partial \mathbb{D}$, and π is **horizontally trivial**, which we take to mean the existence of a tubular neighbourhood $\nu_E(\partial_h E)$ of $\partial_h E$ and a trivialization ϕ so that

$$\begin{tikzcd} \nu_E(\partial_h E) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & \nu_E(\partial E_z) \times \mathbb{D} \arrow[dl, "\mathrm{pr}_2"] \\ & \mathbb{D} & \end{tikzcd}$$

commutes, where E_z is a regular reference fiber of π. Furthermore, the map ϕ should provide the following identification of Ω:

$$(\phi^{-1})^* \Omega \stackrel{!}{=} \Omega|_{TE_z} + K \pi^* \omega_b,$$

for some $K > 0$ and ω_b some symplectic form on $\mathbb{D}$.

An **exact symplectic Lefschetz fibration** is a symplectic Lefschetz fibration $(E, \pi, \Omega = d\lambda, J, j)$ so that the closed 2-form $\Omega \in \Omega^2(E)$ is exact.

Seidel's monograph [Sei08] formulates the theory for exact Lefschetz fibrations. Since we are mainly concerned with strong symplectic fillings obtained through Lefschetz fibrations (recall that the symplectic form ω on a symplectic filling (W, ω) of some contact manifold must be exact near the boundary), we shall do the same. The results also hold for non-exact symplectic Lefschetz fibrations, however: the results from *fibered* Picard-Lefschetz theory discussed in Section 2.3 are formulated in the non-exact setting, and specialise to the results for the Lefschetz case discussed here.

Remark 2.2.2. By the complex Morse lemma [Arn+98], the condition that there exist integrable complex structures J and j near the critical points and values, respectively, in such a way that π is (J, j)-holomorphic wherever they are defined implies the existence of Lefschetz charts (U, φ) and (V, ψ) near E^{crit}, so that

$$\tau \circ \pi \circ \sigma^{-1} : (z_1, \ldots, z_n) \mapsto z_1^2 + \ldots + z_n^2, \tag{2.1}$$

just as in Definition 2.1.1. Conversely, complex charts as above give rise to integrable almost complex structures near E^{crit}.

Just as topological Lefschetz fibrations are locally trivial fiber bundles away from the critical points, symplectic Lefschetz fibrations are *symplectic fiber bundles* on $E \setminus E^{\text{crit}}$:

Definition 2.2.3. A **symplectic fiber bundle** (E, π, Ω) consists of a manifold E equipped with a closed 2-form $\Omega \in \Omega^2(E)$ and a fiber bundle $\pi : E \to S$ over a smooth surface S, such that Ω restricted to any fiber of π is nondegenerate.

Indeed, condition (ii) in Definition 2.2.1 implies that symplectic Lefschetz fibrations are symplectic fiber bundles away from E^{crit}: let γ be a curve which lies entirely in the smooth part of the fiber E_z, then $\pi \circ \gamma \equiv z$, so that $D\pi[\gamma'] = 0$. Hence $T_x E_z \subset \ker(D\pi(x))$ for all $x \in E_z$, which means that Ω is nondegenerate on all fibers.

2.2.1 Symplectic Parallel Transport

Recall that in order to define the monodromy of a topological Lefschetz fibration in Section 2.1.3, we resorted to a choice of Ehresmann connection on the fiber bundle $E \backslash \pi(E^{\text{crit}})$ and set the monodromy along a loop γ to be parallel transport along this loop. It turns out that symplectic fibre bundles come with a canonical notion of symplectic parallel transport, which we use to define the monodromy in the same way, as well as the vanishing cycles of a symplectic Lefschetz fibration. We follow [WW16] in doing so.

Proposition 2.2.4 ([WW16, p. 7])**.** *Let (E, π, Ω) be a symplectic fiber bundle over a surface S. The distribution $\mathcal{H}_\Omega$ of TE defined by*

$$\mathcal{H}_{\Omega,x} := (\ker(D\pi(x)))^\Omega \subset T_x E$$

is an Ehresmann connection.

Proof. To prove that $\mathcal{H}_\Omega$ defines a connection, we need to show that $D\pi(x)$ maps $\mathcal{H}_{\Omega,x}$ isomorphically onto $T_{\pi(x)}S$. So let $v \in \ker D\pi(p) \cap \mathcal{H}_\Omega$. The tangent vector v being in the symplectic complement of the vertical bundle, we have

$$\Omega(v,u) = 0, \qquad \text{for all } u \in T^v_x E.$$

However, v itself is in the vertical bundle, on which Ω was assumed to be nondegenerate, so $v = 0$. As Ω is nondegenerate on the vertical bundle, the subspace $\mathcal{H}_{\Omega,x}$ is two dimensional like $T_{\pi(x)}S$, so $D\pi(x)|_{H_{\Omega,x}}$ is an isomorphism. □

We refer to this connection as a *symplectic connection* for reasons explained by the next lemma.

Lemma 2.2.5 ([WW16, p. 7]). *Let $\pi : (E,\pi) \to S$ be a symplectic fiber bundle and $\gamma : [a,b] \to S$ be a path in the base. Then the parallel transport maps associated to the connection $\mathcal{H}_\Omega$*

$$\rho_\gamma : (E_{\gamma(a)}, \Omega|_{TE_{\gamma(a)}}) \to (E_{\gamma(b)}, \Omega|_{TE_{\gamma(b)}})$$

are symplectomorphisms.

Proof. This follows from an observation on horizontal vector fields with respect to the symplectic connection. If $V \in \mathfrak{X}(E)$ is any horizontal vector field and E_z is any fiber, then

$$\mathcal{L}_V \Omega|_{TE_z} = (d\imath_V \Omega)|_{TE_z} = d(\imath_V \Omega|_{TE_z}) = 0,$$

since the tangent spaces TE_z lie in $\ker(D\pi)$, so by definition of $\mathcal{H}_\Omega$, $\imath_V \Omega|_{TE_z} \equiv 0$.

This implies that the flow of V preserves the restriction of Ω to the fibers, and as parallel transport is defined as the flow of the horizontal vector field X_γ, this finishes the proof. □

We now extend symplectic parallel transport to symplectic Lefschetz fibrations (E,π,Ω,J,j). On $E \setminus E^{\text{crit}}$, π is an ordinary symplectic fibration, so parallel transport is well-defined for any path with image in $S \setminus \pi(E^{\text{crit}})$. We extend parallel transport to singular fibers. Let $\gamma : [0,1] \to S$ be an embedded path so that $\gamma(1) = x_0 \in \pi(E^{\text{crit}})$ and $\gamma([0,1)) \subset S \setminus \pi(E^{\text{crit}})$. Such a path is called a **vanishing path**.

Then parallel transport extends to a continuous map

$$\rho_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}, \quad x \mapsto \lim_{t \nearrow 1} \rho_{\gamma|_{[0,t]}}(x).$$

For more details on this construction, see [Sei03, Lemma 1.13].

2.2.2 Symplectic Vanishing Cycles and Monodromy

For a topological Lefschetz fibration (E, π) with regular fiber F, we used the local coordinate description near $x_0 \in E^{\text{crit}}$ given by Lefschetz charts to define the vanishing cycle corresponding to x_0. We concluded that the singular fiber over $\pi(x_0)$ can be obtained from F by collapsing the vanishing cycle.

It is this property that motivates the definition of vanishing cycles in the symplectic setting as those points in a given fiber that map to the critical point of interest under symplectic parallel transport.

Definition 2.2.6. Suppose (E^{2n+2}, π, Ω) is an exact symplectic Lefschetz fibration over S. To any vanishing path γ, we associate its **vanishing thimble** defined by

$$T_\gamma = \left\{ x \in \bigcup_{t \in [0,1)} E_{\gamma(t)} \mid \lim_{t_0 \nearrow 1} \rho_{\gamma|_{[t,t_0]}}(x) = x_0 \right\} \cup \{x_0\}.$$

The **vanishing cycle** associated to γ is defined to be

$$C_\gamma := \partial T_\gamma = T_\gamma \cap E_{\gamma(0)}.$$

In his extensive monograph on symplectic Lefschetz fibrations [Sei08, (16b)], Seidel explains that the vanishing thimble $T_\gamma \subset E^{2n+2}$ is a Lagrangian submanifold of the total space diffeomorphic to an $(n+1)$-ball, and the vanishing cycle $C_\gamma \subset E^{2n}_{\gamma(0)}$ is a Lagrangian n-sphere in the fiber. Note in particular that the Dehn twist $\tau_{C_\gamma} \in \operatorname{Symp}(E_{\gamma(0)})$ is well-defined.

As for the monodromy of symplectic Lefschetz fibrations, we first make the following observation.

Proposition 2.2.7. *The monodromy of a symplectic Lefschetz fibration is isotopic to a symplectomorphism.*

Proof. Recall from Section 2.1.3 that to define the monodromy μ_γ of any fibration along a loop γ in the disk $\mathbb{D}$, one chooses an Ehresmann connection on the smooth part of the total space and sets $\mu_\gamma = \rho_\gamma$. In the case of symplectic Lefschetz fibrations, the canonical symplectic connection yields a parallel transport system consisting of symplectomorphisms by Lemma 2.2.5, which proves the claim. □

In fact, we have

Theorem 2.2.8 ([Sei03, Proposition 1.15], [Sei08, (16c)])**.** *Let $\gamma : [0,1] \to S$ be a vanishing path, and ℓ be a loop in S "doubling" γ as in Figure 2.1, winding anticlockwise around $\gamma(1)$. Then the monodromy of π along ℓ is symplectically isotopic to the Dehn twist along the vanishing cycle C_γ:*

$$[\mu_\ell] = [\tau_{C_\gamma}] \in \operatorname{Symp}(E_{\gamma(0)}).$$

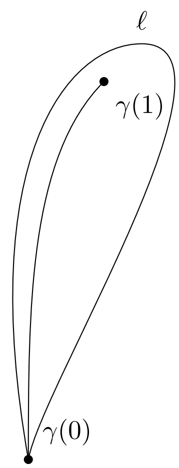

Figure 2.1: The loop ℓ obtained by doubling γ

Therefore, any critical point of an exact symplectic Lefschetz fibration gives rise to an embedded Lagrangian sphere in the regular fiber F, and establishes the Dehn twist as an element of its symplectic mapping class group.

Conversely, given any embedded Lagrangian sphere C in an exact symplectic manifold (F, ω), one can construct an exact symplectic Lefschetz fibration over the unit disk $S = \mathbb{D}$ whose regular fibers are symplectomorphic to (F, ω), and which has C as its only vanishing cycle [Sei08, (16e)].

This result extends to multiple critical values:

Theorem 2.2.9 ([Sei08, (16c), (16e)]). *For an exact symplectic Lefschetz fibration $\pi : (E, \Omega) \to \mathbb{D}$ with multiple critical values and a corresponding collection of vanishing paths $(\gamma_1, \ldots, \gamma_k)$ intersecting only a common starting point $* \in F$, the monodromy along $\partial\mathbb{D}$ is symplectically isotopic to*

$$\tau_{C_{\gamma_1}} \circ \cdots \circ \tau_{C_{\gamma_k}}.$$

On the other hand, given a collection of embedded Lagrangian spheres $(C_1, \ldots, C_k)$ in an exact symplectic manifold (F, ω), there is an exact symplectic Lefschetz fibration over the unit disk with regular fibers symplectomorphic to (F, ω) and whose collection of vanishing cycles is given by $(C_1, \ldots, C_k)$.

2.3 Symplectic Lefschetz-Bott Fibrations

We generalize the results of Section 2.2 to Lefschetz-Bott fibrations, which can be considered as Lefschetz fibrations whose critical locus is a smooth submanifold of the total space instead of a discrete subset.

The related theory is also known as *fibered Picard-Lefschetz theory*, and was mostly known to Seidel circa 1998 (unpublished notes). A first comprehensive reference was given by Perutz [Per07], which largely shares the structure of Seidel's [Sei03].

For our purposes, it suffices to consider Lefschetz-Bott fibrations over $\mathbb{C}$, although one could define them over any surface S as we did for Lefschetz fibrations. We use the definition given in [Oba20, Section 3.2].

As mentioned, the main generalization in sympletic Lefschetz-Bott fibrations from symplectic Lefschetz fibrations consists in allowing the critical locus E^{crit} to be a smooth submanifold, which requires a suitable modification of the non-degeneracy condition for the complex Hessian. To this end, we need a piece of vocabulary:

Definition 2.3.1. Let W^{2n} be a smooth manifold, equipped with an almost complex structure J and a closed 2-form Ω. Let N be an almost complex submanifold of (W, J). The form Ω is said to be **normally Kähler** near N if there exists a tubular neighbourhood $\nu_M(N)$ of N in W which can be foliated by normal slices $\{D_x\}_{x\in N}$ so that $J|_{TD_x}$ is integrable and $\Omega|_{TD_x}$ is J–Kähler for each $x \in N$.

According to [Per07], this is a technical convenience that could most likely be shown to always be satisfied after a perturbation of J and Ω. We now state the definition of symplectic Lefschetz-Bott fibrations in full for the convenience of the reader, though one should note that only the conditions on E^{crit} and the Hessian differ from Definition 2.2.1.

Definition 2.3.2. A **symplectic Lefschetz-Bott fibration** is a tuple (E, π, Ω, J, j) consisting of

- an even dimensional manifold E;
- a smooth proper map $\pi : E \to \mathbb{C}$ whose critical points E^{crit} lie in $\text{int}(E)$;
- a closed 2-form Ω on E;
- an almost-complex structure J defined on a neighbourhood of $E^{\text{crit}} \subset E$;
- a complex structure j on a neighbourhood of the critical values in $\mathbb{C}$ compatible compatible with the standard orientation.

These are subject to the following conditions:

(i) π is (J, j)-holomorphic near E^{crit} where J and j are defined;

(ii) E^{crit} is a smooth submanifold of E with finitely many connected components;

(iii) Ω is nondegenerate on the vertical bundle $T^vE = \ker D\pi$;

(iv) Near E^{crit} where J is defined, Ω is nondegenerate, compatible with J, and normally Kähler with respect to J;

(v) The complex normal Hessian $D^2\pi_x|_{TD_x \otimes TD_x}$ is nondegenerate for all $x \in E^{\text{crit}}$, where D_x is a normal slice of a tubular neighbourhood of E^{crit}.

Again, we are mainly interested in the case where the base is the unit disk $\mathbb{D}$ and regular fibers have nonempty boundary. In this case, we additionally impose items (iv) and (v) from Definition 2.2.1.

The complex structures J and j will occasionally be suppressed from the notation, so that we refer to symplectic Lefschetz-Bott fibrations by (E, π, Ω) or by $\pi : (E, \Omega) \to \mathbb{C}$.

When we are interested only in topological properties of Lefschetz-Bott fibrations, notably in section Section 5.6, where we distinguish a collection of symplectic Lefschetz-Bott fibrations, we use the notion of a **topological Lefschetz-Bott fibration**. A topological Lefschetz-Bott fibration is a tuple (E, π, Ω, J, j) for which E, π, J, and j satisfy the same conditions as a symplectic Lefschetz-Bott fibration, but where Ω is only required to be a closed 2-form defined in a neighbourhood of E^{crit}. Item (iii) and horizontal triviality will no longer be required. We explained in Section 2.2.1 how $\ker(D\pi(x))^{\Omega}$ defines a canonical connection on E if Ω is global, so in the case of topological Lefschetz-Bott fibrations, instead of item (iv) from Definition 2.2.1, we require $\mathcal{H}_x \subset T_x\partial_h E$ for all $x \in \partial_h E$, where $\mathcal{H}$ is a chosen Ehresmann connection.

Remark 2.3.3. By the parametric version of the holomorphic Morse lemma [Arn+98], for a topological Lefschetz fibration (E^{2n}, Ω, π), there exist charts (U, σ) on E near each critical point $x_0 \in E^{\text{crit}}$ and (V, τ) near each critical value on $\mathbb{C}$ in which we have

$$\tau \circ \pi \circ \sigma^{-1}(z_1, \dots, z_n) = \sum_{j=1}^{k} z_j^2,$$

where k is the corank of $D\pi(x_0)$ (or the codimension of $E^{\text{crit}} \subset E$).

2.3.1 Vanishing Cycles

In our study of the singular fibers of Lefschetz fibrations, vanishing cycles have always consisted of a subset of the regular fiber; the singular fiber is then obtained by simply collapsing the vanishing cycle in the topological case, or by parallel transporting it along a vanishing path in the symplectic setting. For Lefschetz-Bott fibrations, the singular fibers are also obtained by contracting the corresponding vanishing cycle, although the contraction will no longer be to a single point.

Note that symplectic Lefschetz-Bott fibrations are symplectic fiber bundles in the sense of Definition 2.2.3 away from E^{crit}. The parallel transport maps ρ_γ along paths γ in $\mathbb{C}$ can be extended to parallel transport along a vanishing path just as we did for Lefschetz fibrations [Per07, Section 2.3.1].

Definition 2.3.4. Let $\pi : (E, \Omega) \to \mathbb{D}$ be a symplectic Lefschetz-Bott fibration over the unit disk and $\gamma : [0,1] \to \mathbb{D}$ be a vanishing path to a critical value of π. To any connected component N of $E^{\text{crit}} \cap E_{\gamma(1)}$, associate the **vanishing thimble**

$$T_{\gamma,N} = \left\{ x \in \bigcup_{t \in [0,1)} E_{\gamma(t)} \mid \lim_{t_0 \nearrow 1} \rho_{\gamma|_{[t,t_0]}}(x) \in N \right\} \cup N.$$

Define the **vanishing cycle** associated to γ by

$$C_\gamma = \partial T_{\gamma,N} = T_{\gamma,N} \cap E_{\gamma(0)}.$$

The following lemma provides some intuition on the structure of the vanishing cycles.

Lemma 2.3.5 ([Per07, Lemma 2.5])**.** *C_γ is a smooth coisotropic submanifold of $E_{\gamma(0)}$, and the restriction*

$$\rho_\gamma : C_\gamma \to N$$

is a smooth fiber bundle with spheres S^k as fibers, where k is the rank of $D\pi$. The structure group of $C_\gamma \overset{\rho_\gamma}{\to} N$ can be reduced in a canonical way to $O(k+1)$.

The vanishing cycle C_γ consists of those points in $E_{\gamma(0)}$ for which the limit parallel transport map ρ_γ is defined and lands in $N \subset E^{\text{crit}} \cap E_{\gamma(1)}$. The singular fiber $E_{\gamma(1)}$ can hence be seen to be obtained from a regular fiber $E_{\gamma(0)}$ by applying ρ_γ, which may be thought of as a deformation retract of the vanishing cycle C_γ to the submanifold N.

[Per07, p. 782] provides a discussion on how Lemma 2.3.5 gives C_γ the structure of a *spherically fibered coisotropic submanifold* of the regular fiber $E_{\gamma(0)}$. The significance of this result is that a generalization of the Dehn twist, called a *fibered Dehn twist*, reviewed in the next section, can be defined along any spherically fibered coisotropic.

2.3.2 Fibered Dehn Twists as Monodromy Maps

Following [CDK14, Section 2], we describe a model situation of the fibered Dehn twist. This time, the model is a contact manifold (P, α) whose Reeb orbits are periodic, so that the flow of the Reeb vector field R_α defines a right S^1-action on P. Note that in particular the Boothby-Wang bundles over integral symplectic manifolds encountered in Definition 4.1.1 satisfy this condition. Choose a function

$$f : [0,1] \to \mathbb{R}$$

which is constantly equal to 2π in a neighbourhood of 0 and equal to 0 in a neighbourhood of 1. Consider now (a part of) the symplectization of (P, α)

$$(P \times [0,1], d(e^t \alpha)).$$

On this domain, we construct a diffeomorphism equal to the identity near the boundary by setting

$$\psi : (x,t) \mapsto (x \cdot f(t) \bmod 2\pi, t).$$

The fact that $\psi = \mathrm{id}$ near the boundary follows from the choice of f. This is in fact a symplectomorphism: let ψ_t with a subscript denote the flow of the Reeb vector field R_α. Then

$$\frac{d}{dt}\psi^*(e^t\alpha) = \frac{d}{dt}\psi^*_{f(t)}(e^t\alpha) = \psi^*_{f(t)}(\mathcal{L}_{f(t)R_\alpha}(e^t\alpha) + e^t\alpha).$$

The Lie derivative evaluates to

$$\begin{aligned}
\mathcal{L}_{f(t)R_\alpha} &= d\imath_{f(t)R_\alpha}(e^t\alpha) + \imath_{f(t)R_\alpha} d(e^t\alpha) \\
&= d(f(t)e^t) + \imath_{f(t)\alpha}(e^t dt \wedge \alpha + d^t d\alpha) \\
&= d(f(t)e^t) - f(t)e^t dt \\
&= -d\left(-f(t)e^t + \int_0^t f(s)e^s ds + A\right),
\end{aligned}$$

where A is some integration constant. Thereby,

$$\psi^*(e^t\alpha) = e^t\alpha - d\left(-f(t)e^t + \int_0^t f(s)e^s ds + A\right),$$

which implies that ψ is a symplectomorphism.

Definition 2.3.6. Suppose that (W, ω) is a symplectic manifold with convex boundary such that ∂W admits a contact form whose Reeb orbits are periodic. Then we may identify a collar neighbourhood with $(P \times [0,1], d(e^t\alpha))$, where $(P = \partial W, \alpha)$ is a contact manifold just as above. Define a symplectomorphism $\tilde{\psi}$ of W by setting it to be ψ on the collar neighbourhood and the identity on the rest of W. The map $\tilde{\psi}$ is called a **right-handed fibered Dehn twist** along ∂W.

Remark 2.3.7. In fact, it is possible to define a fibered Dehn twist along any spherically fibered coisotropic submanifold $C \subset W$ ([Per07], [WW16]). A simple case of a spherically fibered coisotropic is that of a Lagrangian sphere L, in which case a fibered Dehn twist along L reduces to a Dehn twist along L.

It is sometimes possible to establish relations between Dehn and fibered Dehn twists. A particular result we will make use of in section Section 5.5, where we construct distinct fillings of the A_k-type singularity, describes the fibered Dehn twist along the boundary of a particular class of symplectic manifolds. Set

$$V_d(\delta) = \left\{ (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid \sum_{j=0}^n z_j^d = 1 \right\} \cap \left\{ \sum_{j=0}^n |z_j|^2 \leq \delta^2 \right\}.$$

Theorem 2.3.8 ([AA16, Theorem 1.1]). *Let ω_0 be the symplectic form on $V_d(\delta)$ given by restricting the standard form on $\mathbb{C}^{n+1}$. With respect to this symplectic structure, a fibered Dehn twist along $\partial V_d(\delta)$ is symplectically isotopic to the product of $d(d-1)^{n+1}$ right-handed Dehn twists.*

Fibered Dehn twists can be realized as monodromy maps of symplectic Lefschetz-Bott fibrations:

Theorem 2.3.9 ([Per07, Monodromy Theorem 2.16]). *Let $\pi : (E, \Omega) \to \mathbb{D}$ be a symplectic Lefschetz-Bott fibration with a single critical value in $\operatorname{int}(\mathbb{D})$ and γ a corresponding vanishing path. Then the monodromy along the loop obtained by doubling γ based is symplectically isotopic to a fibered Dehn twist along C_γ, denoted by τ_{C_γ}.*

Moreover, an existence statement holds:

Proposition 2.3.10 ([WW16, Proposition 2.13]). *Let (M, ω) be a symplectic manifold and $C \subset M$ a spherically fibered coisotropic submanifold of M. Then there exists a symplectic Lefschetz-Bott fibration $\pi : (E, \Omega) \to \mathbb{C}$ with a single critical value whose fibers are symplectomorphic to (M, ω) and whose monodromy is symplectically isotopic to a fibered Dehn twist along C.*

Note that by scaling $\mathbb{C}$, we may assume this Lefschetz-Bott fibration takes values in $\mathbb{D}$.

Similar results hold for multiple critical values, just as in the case of Lefschetz fibrations [Sei08, (16e)].

Chapter 3

Open Book Decompositions

To motivate this chapter, let us have a look at the boundary of the total space of a Lefschetz-Bott fibration. Recall that a topological Lefschetz-Bott fibration (E, π, J, j) with fiber F over $\mathbb{D}$ is assumed to admit a decomposition of its total space as $\partial E = \partial_v E \cup \partial_h E$, where

- $\partial_v E := \pi^{-1}(\partial\mathbb{D})$; since $\pi|_{\partial_v E}$ is a locally trivial S^1-bundle with fiber F, it is a mapping torus whose gluing is ψ, the monodromy of the Lefschetz-Bott fibration. Denote this mapping torus by $\partial_v E = F(\psi)$.
- $\partial_h E := \bigsqcup_{z\in\mathbb{D}} \partial(\pi^{-1}(z))$; as $\pi|_{\partial_h E}$ has no critical values, it defines a fiber bundle over $\mathbb{D}$, which is contractible, and hence $\partial_h E \cong \partial F \times \mathbb{D}$ is trivial.

Both boundary components meet in the codimension two corner given by

$$\partial(\partial_v E) = \bigsqcup_{z\in\partial\mathbb{D}} \partial(\pi^{-1}(z)) = \partial(\partial_h E).$$

Abstractly, this corner is diffeomorphic to $\partial F \times S^1$. Hence the boundary of the total space can be written as

$$\partial E = F(\psi) \cup_{\partial F\times S^1} (\partial F \times \mathbb{D}).$$

Starting from any manifold F and a diffeomorphism $\psi \in \mathrm{Diff}(F)$ which is the identity near ∂F, the same gluing procedure yields a new manifold $OB(F;\psi)$; the pair (F, ψ) is known as an *abstract open book*.

In Section 3.1, we study general properties open books, before seeing when and how $OB(F;\psi)$ can be endowed with a contact structure in Section 3.2. Finally, in Section 3.3, we return to Lefschetz-Bott fibrations to see how the total space of a symplectic Lefschetz-Bott fibration with regular fiber F and monodromy ψ can act as a strong (or Stein, in the Lefschetz case) symplectic filling of the contact manifold $OB(F;\psi)$.

3.1 Abstract Open Books and Open Book Decompositions

In this section, we first introduce the two flavours open books come in, and then explain in Section 3.1.1 how to move between these notions. We finish the section by examining different open book structures on S^3. For an excellent and thorough introduction to the subject, the author recommends [Etn05], although we mainly follow [Gei08, Chapter 7] in this exposition.

Abstract Open Books

Definition 3.1.1. An **abstract open book** is a pair (F, ψ), where

- F is an oriented compact manifold of dimension $2n$ with boundary, and
- $\psi : F \to F$ is a diffeomorphism which is equal to the identity near ∂F.

The diffeomorphism ψ is called the **monodromy**, and F is called the **page** of the abstract open book.

Given an abstract open book with page F and monodromy ψ, we can construct a $(2n+1)$-manifold $OB(F;\psi)$: first, define the **mapping torus**

$$F(\psi) := F \times [0, 2\pi]/((x, 2\pi) \sim (\psi(x), 0)).$$

This is a manifold of dimension $2n + 1$ whose boundary is $\partial F \times S^1$. Note also that there is a natural fibration over S^1 given by

$$[x, \varphi] \mapsto \varphi.$$

Next, consider $\partial F \times \mathbb{D}$. This is also a $(2n+1)$-manifold with boundary $\partial F \times S^1$, so we can glue these manifolds together at their common boundary by the identity map and set

$$OB(F;\psi) := F(\psi) \cup_{\partial F \times S^1} (\partial F \times \mathbb{D}). \tag{3.1}$$

Remark 3.1.2. Note that $OB(F;\psi)$ has no boundary. Moreover, gluing along the boundary will not produce a smooth manifold in general. Therefore, one should instead glue collar neighbourhoods of the boundary when the smooth structure of $OB(F;\psi)$ is relevant in applications, as in Theorem 3.2.1 below, where we will endow $OB(F;\psi)$ with a contact structure.

The manifold $OB(F;\psi)$ comes with a natural fiber bundle over S^1: on $F(\psi)$, we can take the obvious fibration $[x, \varphi] \mapsto \varphi$ from before, but now we need to extend this to $\partial F \times \mathbb{D}$. Letting $i : \partial F \times \mathbb{D} \hookrightarrow OB(F;\psi)$ be the embedding obtained by the inclusion into $F(\psi) \sqcup (\partial F \times \mathbb{D})$ followed by the quotient projection, set

$$B := i(\partial F \times \{\mathbf{0}\}).$$

Then we may define the bundle

$$p : OB(F;\psi) \setminus B \to S^1$$

defined for $[x, \varphi] \in F(\psi)$ by

$$p([x, \varphi]) = \varphi,$$

and for $[x, re^{i\varphi}] \in \partial F \times \mathbb{D}$ by

$$p([x, re^{i\varphi}]) = \varphi.$$

The fibers of p are seen to be

$$p^{-1}(\varphi) = \{[x, \varphi] \mid x \in F\} \cup_{\partial F \times S^1} \{(y, re^{i\varphi}) \mid y \in \partial F,\, r \in (0, 1]\} \cong \operatorname{int}(F),$$

and they satisfy $\partial\overline{p^{-1}(\varphi)} = i(\partial F \times \{\mathbf{0}\}) = B$.

Open Book Decompositions

Open book decompositions place more emphasis on the S^1-bundle structure, like the one just constructed.

Definition 3.1.3. An **open book decomposition** of a manifold M is a pair (B, p) consisting of

- a codimension two submanifold B with trivial normal bundle in M called the **binding** of the decomposition, and
- a smooth fiber bundle $p : M \setminus B \to S^1$.

We further require that B have a trivial tubular neighbourhood $B \times \mathbb{D}$ on which p is the projection to the angular coordinate of the $\mathbb{D}$-factor. The fiber $p^{-1}(\varphi)$ is called the **page** of the open book decomposition.

The pages $p^{-1}(\varphi)$ are codimension one submanifolds of $M \setminus B$ without boundary, and as on $B \times \mathbb{D}$, we have $p(x, re^{i\varphi}) = \varphi$, we see that

$$p^{-1}(\varphi) \cap (B \times \mathbb{D}) = \{(x, re^{i\varphi}) \mid x \in B, r \in (0, 1)\}.$$

The closure of the page in M is thus a codimension one submanifold with boundary B.

Remark 3.1.4. The fibration $p : OB(F; \psi) \setminus B \to S^1$ constructed after Equation (3.1) defines an open book decomposition (B, p) on $OB(F; \psi)$.

Remark 3.1.5. In principle, one can define both abstract open books (F, ψ) and open book decompositions (B, p) on a manifold M without any constraints on $\dim(M)$ or $\dim(F)$. For the contact geometric setting of this text, only the following cases are relevant:

- manifolds M or $OB(F; \psi)$ admitting open book decompositions are of dimension $2n + 1$;
- the pages F of abstract open books and the pages $\overline{p^{-1}(\varphi)}$ of open book decompositions have dimension $2n$;
- the binding B has dimension $2n - 1$.

3.1.1 From an Open Book to an Abstract Open Book

Given an abstract open book (F^{2n}, ψ), we have seen how the $(2n+1)$-manifold $OB(F;\psi)$ defined in Equation (3.1) admits an open book decomposition with binding $B^{2n-1} = i(\partial F \times \{\mathbf{0}\})$ and fiber bundle $p : OB(F;\psi) \setminus B \to S^1$. Conversely, given an open book decomposition, we may recover the abstract open book as follows:

Construction 3.1.6 ([Gei08, p. 150])**.** Let (B, p) be an open book decomposition of M. Define F as the intersection of any page, for example $p^{-1}(1)$, with the complement of an open tubular neighbourhood $B \times \mathrm{Int}(\mathbb{D}^2_{\frac{1}{2}})$. Choose a Riemannian metric on M and a vector field on M we shall call ∂_φ such that

- ∂_φ is orthogonal to the pages;
- $Dp[\partial_\varphi] = \frac{\partial}{\partial\theta} \in \mathfrak{X}(S^1)$, where we denote the coordinate on S^1 by θ;
- ∂_φ vanishes on B.

Letting ψ_t be the flow of ∂_φ, set $\psi := \psi_{2\pi}$. Then (F, ψ) is an abstract open book such that $OB(F;\psi)$ is diffeomorphic to M.

Before verifying the consistency of this procedure, note that this allows us to speak of open book decompositions and abstract open books more or less interchangeably; the two concepts are not quite equivalent, however, as abstract open books are merely defined up to diffeomorphism, whereas we can consider open book decompositions up to isotopy. Moreover, we have defined abstract open books for compact pages F, so that also the manifold $OB(F;\psi)$ is compact, whereas we did not require this for manifolds on which open book decompositions can be defined.

Claim 3.1.7. *Such a vector field $\partial_\varphi \in \mathfrak{X}(M)$ exists.*

Proof. We start by taking the coordinate vector field ∂_φ on the tubular neighbourhood $B \times (\mathbb{D} \setminus \{\mathbf{0}\})$. Write $\mathbb{D}^\times$ for $\mathbb{D} \setminus \{\mathbf{0}\}$. ∂_φ satisfies $Dp[\partial_\varphi] = \frac{\partial}{\partial\theta}$. Note that ∂_φ is transverse to the pages: suppose it were not, then at some point, ∂_φ would be tangent to a page and there would be a path in the page to that point whose velocity vector is ∂_φ. But along this path, p is constant, and so $Dp[\partial_\varphi] = 0$ at this point, which is a contradiction to $Dp[\partial_\varphi] = \frac{\partial}{\partial\theta}$. Hence as ∂_φ is transverse to the pages, it defines a nonzero section of $T(B \times \mathbb{D}^\times)/Tp^{-1}(\varphi) \cong \mathbb{R}$, and we may choose a metric which identifies this quotient with the orthogonal complement of $Tp^{-1}(\varphi)$. We may extend this metric to all of M by a partition of unity argument in such a way that $TM/Tp^{-1}(\varphi) \cong Tp^{-1}(\varphi)^\perp$.

Outside of the tubular neighbourhood $B \times \mathbb{D}$, extending ∂_φ by any smooth section of $T(B \times \mathbb{D}^\times)/Tp^{-1}(\varphi) \cong \mathbb{R}$ yields a vector field on $M \setminus B$ orthogonal to the pages.

To achieve that this extension satisfy $Dp[\partial_\varphi] = \frac{\partial}{\partial\theta}$, we construct an extension as follows. Let $U \subset S^1$ a domain for a bundle chart $\alpha : p^{-1}(U) \to F$, where F

denotes the abstract fiber. Recall that this means that $(p, \alpha) : p^{-1}(U) \to U \times F$ is a diffeomorphism. Fixing $q \in F$, define

$$s_\alpha^q : S^1 \to M \setminus B, \qquad s_\alpha^q(\varphi) := (p, \alpha)^{-1}(\varphi, q).$$

We have $p \circ s_\alpha^q(\theta) = \theta$, so the $(s_\alpha^q)_{q \in F}$ form a smooth family of sections foliating $p^{-1}(U)$; the fact that s_α^q is a section means that

$$\frac{\partial}{\partial \theta} = D(p \circ s_\alpha^q)(\theta)[\frac{\partial}{\partial \theta}] = Dp(s_\alpha^q(\theta)) Ds_\alpha^q(\theta)[\frac{\partial}{\partial \theta}].$$

In view of this, for any $x \in p^{-1}(U)$ there exists a unique $q \in F$ and a unique $\theta \in U \subset S^1$ such that $x = s_\alpha^q(\theta)$, which leads us to define ∂_φ on the set $p^{-1}(U)$ as

$$\partial_{\varphi,U}(x) := Ds_\alpha^q(\theta)[\frac{\partial}{\partial \theta}], \qquad x \in p^{-1}(U).$$

As $Dp[\partial_{\varphi,U}] = \frac{\partial}{\partial \theta}$, we may conclude as for ∂_φ on the tubular neighbourhood that $\partial_{\varphi,U}$ is transverse to the pages, and, if necessary, adjust the Riemannian metric on this coordinate patch to ensure it is orthogonal. Now let $\mathcal{U}$ be a cover of S^1 by bundle chart domains and let λ_U, $U \in \mathcal{U}$, a partition of unity subordinate to this cover. Set

$$\partial_\varphi := \sum_{U \in \mathcal{U}} \lambda_U \partial_{\varphi,U},$$

which is still orthogonal to the pages and satisfies

$$Dp[\partial_\varphi] = \sum_U \lambda_U Dp[\partial_{\varphi,U}] = \sum_U \lambda_U \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta}.$$

Lastly, smoothly extend ∂_φ to B by setting $\partial_\varphi|_B = 0$. □

Claim 3.1.8. *The map $\psi := \psi_{2\pi}$, where ψ_t is the flow of ∂_φ, is a diffeomorphism of F (recall that F is the intersection of $p^{-1}(\varphi)$ with the complement of* $\mathrm{Int}(\mathbb{D}_{\frac{1}{2}})$*) which is the identity near the boundary.*

Proof. Compute for $x \in F$

$$\begin{aligned} \frac{d}{dt} p \circ \psi_t(x) &= Dp(\psi_t(x))[\partial_\varphi(\psi_t(x))] \\ &= \frac{\partial}{\partial \theta}(p(\psi_t(x))). \end{aligned}$$

Hence $p \circ \psi_t(x)$ is an integral curve of $\frac{\partial}{\partial \theta}$, whence by uniqueness of integral curves, $p \circ \psi_t(x)$ coincides with the integral curve of $\frac{\partial}{\partial \theta}$ starting at $p(x)$, which can be seen to be $t \mapsto p(x) + t \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. This is 2π-periodic, and thus $p(x) = p(\psi_{2\pi}(x))$, so x and $\psi_{2\pi}(x)$ lie in the same fiber.

On the tubular neighbourhood $B \times \mathbb{D}$, the flow of ∂_φ is just $\psi_t(x, re^{i\varphi}) = (x, re^{i(\varphi+t)})$, which shows that on $F \cap B \times \operatorname{Int}(\mathbb{D})$, which is an open neighbourhood of ∂F in F, ψ is the identity. This also shows that ψ maps F to itself: we already know ψ maps fibers to fibers, and that ψ is a diffeomorphism onto its image, so now that we know that ψ is the identity in the tubular neighbourhood of B, we conclude that no $x \in F$ can be mapped into the part of the tubular neighbourhood with radial coordinate less than $\frac{1}{2}$. This proves the claim. $\square$

This establishes that (F, ψ) is a valid abstract open book to consider.

Claim 3.1.9. *Denote the fibration associated to $OB(F; \psi)$ by p' and the binding $i(\partial F \times \{\mathbf{0}\})$ by B'. Then there is a fiber bundle isomorphism*

$$\begin{tikzcd} OB(F;\psi) \setminus B' \arrow[r] \arrow[d, "p'"] & M \setminus B \arrow[d, "p"] \\ S^1 \arrow[r, "\mathrm{id}"] & S^1 \end{tikzcd}$$

Moreover, $OB(F; \psi)$ and M are diffeomorphic.

Proof. It is clear by construction that F is diffeomorphic to the fibers of p, so that $OB(F; \psi) \setminus B'$ and $M \setminus B$ are fiberwise diffeomorphic. Denote this diffeomorphism by η_φ. Then sending $(x, \varphi) \in \bigsqcup_{\varphi \in S^1} F_\varphi \cong OB(F; \psi) \setminus B'$ to $(\eta_\varphi(x), \varphi)$ is the required fiber bundle isomorphism.

The fact that M and $OB(F; \psi)$ are diffeomorphic now follows from the fact that the boundaries of the pages in M and $OB(F; \psi)$ are B and B', respectively, which are diffeomorphic since $B \cong \partial(\overline{p^{-1}(\varphi)}) \cong \partial F \cong i(\partial F \times \{\mathbf{0}\}) = B'$. $\square$

Observe that this also proves independence (up to diffeomorphism) of all the choices made in the construction of ∂_φ: namely, any vector field with the required properties gives rise to an abstract open book (F, ψ) such that $OB(F; \psi) \cong M$.

3.1.2 Examples

Let us give some concrete examples. An easy but useful example of an open book decomposition is what we will refer to as the **standard open book** on $\mathbb{C}$:

Example 3.1.10. On $\mathbb{C}$, there is an open book decomposition with binding $B_0 = \{0\}$ and $p_0 : \mathbb{C}^\times \to S^1$ given by $p_0(z) = \frac{z}{|z|}$. Note that in polar coordinates, $p_0(re^{i\varphi}) = \varphi$. The pages

$$p_0^{-1}(\varphi) = \{re^{i\varphi} \mid r > 0\}$$

are rays emanating from the origin.

This allows us to think of open books as induced by certain fibrations:

Proposition 3.1.11 ([Tor09, Definition 4])**.** *Let $f : M \to \mathbb{C}$ be a smooth function with regular value 0 such that f is transverse to the pages of the standard open book on $\mathbb{C}$. Then setting $B := f^{-1}(0)$ and $p := p_0 \circ f = \frac{f}{|f|}$ defines an open book decomposition of M.*

Open Book Decompositions of S^3

Example 3.1.12. Consider the 3-sphere

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}.$$

1. Define $f : S^3 \to \mathbb{C}$ by $f(z_1, z_2) = z_1$. Using Proposition 3.1.11, there is an open book decomposition of S^3 induced by f with binding

$$B = \{(0, z_2) \in S^3\},$$

and the fibration is

$$\begin{aligned} p : S^3 \setminus B &\to S^1 \subset \mathbb{C} \\ (z_1, z_2) &\mapsto \frac{z_1}{|z_1|}. \end{aligned}$$

Note that in polar coordinates, $p(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2}) = \varphi_1 \in S_1 = \mathbb{R}/2\pi\mathbb{Z}$. The pages are given by

$$p^{-1}(\varphi) = \{(\sqrt{1 - |z_2|^2} e^{i\varphi}, z_2) \in S^3\},$$

which is diffeomorphic to an open unit 2-disk. An arbitrary tangent vector to the pages is

$$\gamma' = -\frac{r_2 \dot{r}_2}{\sqrt{1 - r_2^2}} \partial_{r_1} + \dot{r}_2 \partial_{r_2} + \dot{\varphi}_2 \partial_{\varphi_2}.$$

With the flat metric $g = \sum_i (dx_i)^2$, the coordinate vector fields are orthogonal. In polar coordinates on $\mathbb{C}^2$, this is

$$g = dr_1^{\otimes 2} + r_1^2 d\varphi_1^{\otimes 2} + dr_2^{\otimes 2} + r_2^2 d\varphi_2^{\otimes 2}.$$

Hence one sees that $g(\partial_{\varphi_1}, \gamma') = 0$. Thus ∂_{φ_1} is a vector field orthogonal to the pages, and $Dp[\partial_{\varphi_1}] = \frac{\partial}{\partial \theta}$. Let us compute its flow. In polar coordinates, requiring $\frac{d}{dt}\psi_t(\mathbf{z}) = \partial_{\varphi_1}(\psi_t(\mathbf{z}))$ translates to

$$\frac{d}{dt}\psi_t(\mathbf{z}) = -\frac{r_2 \dot{r}_2}{\sqrt{1 - r_2^2}} \partial_{r_1} + \dot{\varphi_1} \partial \varphi_1 + \dot{r}_2 \partial_{r_2} + \dot{\varphi}_2 \partial_{\varphi_2} \overset{!}{=} \partial_{\varphi_1}.$$

Hence we must have $\psi_t(\mathbf{z}) = (r_1 e^{i(t + \varphi_1)}, z_2)$. The time-$2\pi$ map, our monodromy, is hence the identity. This shows that (B, p) is an open book decomposition of S^3 corresponding to the abstract open book $(\mathbb{D}^2, \mathrm{id})$.

2. Consider a map $f' : S^3 \to \mathbb{C}$, given by $f'(z_1, z_2) = z_1 z_2$. By Proposition 3.1.11, we obtain an open book decomposition whose binding is

$$B' = \{(z_1, z_2) \in S^3 \mid z_1 z_2 = 0\} = \{(z_1, 0) \in S^3\} \cup \{(0, z_2) \in S^3\}.$$

The summands $K_i = \{(z_1, z_2) \in S^3 \mid z_i = 0\}$ for $i = 1, 2$ are called **Hopf links**. The fibration is

$$\begin{aligned} p' : S^3 \setminus B &\to S^1 \subset \mathbb{C} \\ (z_1, z_2) &\mapsto \frac{z_1 z_1}{|z_1 z_2|}, \end{aligned}$$

which in polar coordinates reads

$$(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2}) \mapsto \varphi_1 + \varphi_2.$$

The pages are given by

$$(p')^{-1}(\varphi) = \left\{ (r_1 e^{i\varphi_1}, \sqrt{1 - r_1^2} e^{i(\varphi - \varphi_1)}) \right\}$$

for $(r_1, \varphi_1) \in (0, 1) \times S^1$, which is diffeomorphic to an annulus.

To find the monodromy of this open book, choose a smooth function $\delta : [0, 1] \to [0, 1]$ such that $\delta(r) = 1$ near $r = 0$ and $\delta(r) = 0$ near $r = 1$. Then define the flow

$$\psi_t' : \begin{cases} \varphi_1 \mapsto \varphi_1 + \delta(r_1) \cdot t, \\ \varphi_2 \mapsto \varphi_2 + (1 - \delta(r_1)) \cdot t. \end{cases}$$

Note that $p' \circ \psi_t'(\mathbf{z}) = p'(\mathbf{z}) + t$, implying that this flow is always transverse to the pages. It also evidently maps pages to pages. Note also that

$$\psi_t'(\mathbf{z}) = \delta(r_1)\partial_{\varphi_1} + (1 - \delta(r_1))\partial_{\varphi_2},$$

so that $\phi_t' = \partial_{\varphi_i}$ near $r_i = 0$. Hence we can take a suitable metric so that this is always orthogonal to the pages, and consider $\psi_{2\pi}'$ the monodromy of the open book, which is just a right-handed Dehn twist.

3.2 Contact Structures on Open Books

In this section, we will examine how we can endow the manifold $OB(F; \psi)$ with a contact structure following [Gei08, Section 7.3]. Originally, these results are due to Giroux [Gir02] (see also the translation by Acu [Gir]), partially in collaboration with Mohsen [GM]. . To do so, we require the pages to be Liouville domains $(W, d\lambda)$, and the monodromy to be a symplectomorphism which is the identity near ∂W. From now on, we will include the symplectic structure in the notation for abstract open books, and denote them by $(W, \lambda; \psi)$.

Theorem 3.2.1 ([GM]). *Let $(W, \lambda; \psi)$ be an abstract open book whose pages are Liouville domains $(W, d\lambda)$. Then $OB(W; \psi)$ admits a contact form. We write $OB(W, \lambda; \psi)$ for the resulting contact manifold.*

In the proof, it will be convenient to consider a manifold diffeomorphic to $OB(W; \psi)$ obtained from a generalized version of the mapping torus. Suppose $\eta : W \to \mathbb{R}_+$ is a smooth function which is constant near ∂W. Define the **generalized mapping torus** as

$$W_\eta(\psi) = \{(x, \varphi) \in W \times \mathbb{R} \mid \varphi \in [0, \eta(x)]\}/\sim,$$

where we identify $(x, \eta(x))$ with $(\psi(x), 0)$.

The manifold analogous to $OB(W; \psi)$ obtained through the generalized mapping torus is

$$OB(W, \lambda; \psi) = W_\eta(\psi) \sqcup \partial W \times \mathbb{D}/\sim,$$

glued along the boundary by the identity, which is the manifold we are going to endow with a contact structure. Note that $W_\eta(\psi)$ is diffeomorphic to the usual mapping torus $W(\psi)$ (see [Gei08, Section 7]), and hence the glued manifold using the generalized mapping torus is diffeomorphic to $OB(W; \psi)$.

Before starting the proof, let us make this gluing more precise, and moreover smooth. Suppose η takes the value $c > 0$ near all boundary components. Then we may identify the boundary of $\partial(W_\eta(\psi))$ with

$$\partial(W_\eta(\psi)) = \partial W \times [0, c]/(x, c) \sim (\psi(x), 0) \cong \partial W \times S^1$$

for $S^1 = \mathbb{R}/c\mathbb{Z}$.

This allows us to identify a collar neighbourhood of $\partial(W_\eta(\psi))$ with $\partial W \times [-\epsilon, 0] \times S^1$, for $S^1 = \mathbb{R}/c\mathbb{Z}$. Denote the coordinates in this neighbourhood by (x, s, φ).

Take in turn a collar neighbourhood of $\partial(\partial W \times \mathbb{D}(1 + \epsilon))$ (the slight extension of the radius of the disk is negligible), which we may identify with

$$\partial W \times \mathbb{A}(1, 1 + \epsilon).$$

Here, $\mathbb{A}(1, 1+\epsilon) := \{z \in \mathbb{C} \mid |z| \in [1, 1+\epsilon]\}$. Denote coordinates by $(x, re^{2\pi i\theta})$.

Define the gluing map

$$\begin{aligned} \Phi : \underbrace{\partial W \times \mathbb{A}(1, 1 + \epsilon)}_{\subset \partial W \times \mathbb{D}(1+\epsilon)} &\longrightarrow \underbrace{\partial W \times [-\epsilon, 0] \times S^1}_{\subset W_\eta(\psi)} \\ (x, re^{2\pi i\theta}) &\longmapsto (x, 1 - r, c\theta), \end{aligned}$$

so that as a smooth manifold,

$$OB(W, \lambda; \psi) = W_\eta(\psi) \cup_\Phi (\partial W \times \mathbb{D}(1 + \epsilon)).$$

With this preparation, we can now prove Theorem 3.2.1.

Step 1: Construction of a contact form on the mapping torus

First, we may assume by [Gei08, Lemma 7.3.4] that ψ is an exact symplectomorphism. This means that

$$\psi^*\lambda - \lambda = d\eta$$

for some function η defined up to a constant. By compactness of W, we may assume η only takes positive values. Set

$$\alpha := \lambda + d\varphi \in \Omega^1(W \times \mathbb{R}).$$

Here, φ denotes the coordinate on $\mathbb{R}$. As $d\lambda$ is symplectic, α is contact. It is also invariant under

$$\phi : (x, \varphi) \mapsto (\psi(x), \varphi - \eta(x)) :$$

We have $\phi^*\alpha = \psi^*\lambda + \phi^* d\varphi = \lambda + d\eta + d\varphi - d\eta = \alpha$, and thus α descends to a contact form on the generalised mapping torus $W_\eta(\psi)$. Note that ψ is the identity near the boundary, so η is locally constant near the boundary, hence the generalized mapping torus makes sense to define.

As $(W, d\lambda)$ is a Liouville domain, we may assume there is a symplectic collar of ∂W,

$$\partial W \times [-\epsilon, 0] \to W, \quad (x, s) \mapsto \vartheta_s(x),$$

where ϑ denotes the flow of the Liouville vector field induced by λ. λ may be expressed as $e^s i^*\lambda$, where $i : \partial W \hookrightarrow W$ denotes the inclusion. This collar descends to the mapping torus:

$$j : \partial W \times [-\epsilon, 0] \times S^1 \to W_\eta(\psi), \qquad (x, s, \varphi) \mapsto [\vartheta_s(x), \varphi].$$

Note that the S^1-factor above is still considered as $S^1 = \mathbb{R}/c\mathbb{Z}$.

On this collar, we may write α as $j^*\alpha = e^s i^*\lambda + d\varphi$.

Step 2: Extending α to $\partial W \times \mathbb{D}$

To define a contact form on $\partial W \times \mathbb{D}$, make the ansatz

$$\delta = f(r) i^*\lambda + g(r) d\varphi \in \Omega^1(\partial W \times \mathbb{D}),$$

where $f, g : [0, 1 + \epsilon] \to \mathbb{R}$ are smooth functions. We have to choose f and g appropriately so that δ becomes contact and coincides with α under the identifications made by Φ.

Let us start with compatibility with α. We have that

$$\Phi^*\alpha = \Phi^*(e^s i^*\lambda + d\varphi) = e^{1-s} i^*\lambda + c\, d\varphi.$$

Hence on the collar $\partial W \times \mathbb{A}(1, 1 + \epsilon)$, meaning for $r \geq 1$, we can phrase the compatibility constraints on f and g as follows:

1. $f(r) = e^{1-r}$ and $g(r) = c$ for $r \geq 1$.

Near the centre of $\partial W \times \mathbb{D}$, we prescribe the following form for δ:

2. $f(r) = C_0$ and $g(r) = C_1 r^2$ for $r \leq \epsilon/2$, where $C_0 > 1$ and $C_1 > 0$ are constants.

To ensure that the contact condition is satisfied, a straightforward computation shows

$$\delta \wedge (d\delta)^n = nf^{n-1}(fg' - f'g)i^*\lambda \wedge (di^*\lambda)^{n-1} \wedge dr \wedge d\varphi.$$

Hence for δ to be contact, we need that

3. $(f(r), g(r))$ is never parallel to its tangent vector $(f'(r), g'(r))$ for $r \neq 0$.

For any such choice of functions f and g, we hence obtain a contact structure on the glued manifold $OB(W, \lambda; \psi)$, finishing the proof. $\square$

A manifold admitting an open book decomposition might already be endowed with a contact structure. The following definition provides a notion of compatibility between contact forms and open book decompositions. Before stating it, let us fix the orientation conventions we use for open book decompositions.

Let M be an odd-dimensional oriented manifold with an open book decomposition (B, p), where B also carries an orientation. Orient the pages $p^{-1}(\varphi)$ by requiring that the induced orientation on the boundary of the closure of the pages coincide with the orientation of B. This is equivalent to saying that a basis of the tangent space of the pages is positive if and only if the basis together with ∂_φ, the vector field orthogonal to the pages and vanishing on B, gives a positive basis of M.

Definition 3.2.2. A contact structure ξ on M is said to be **supported by** the open book decomposition (B, p) if there is positive contact form α for ξ such that

1. $d\alpha$ restricted to the tangent space of the pages induces a symplectic form on each page such that the orientation induced by $d\alpha$ coincides with the orientation of the page;

2. α induces a positive contact structure on B.

Such a 1-form is called a **Giroux form**.

Remark 3.2.3. A more concise way of phrasing conditions 1. and 2. in the above definition would be to say that for each page $W_\varphi = p^{-1}(\varphi)$, the manifold $(W_\varphi, d\alpha|_{TW_\varphi})$ has to be a Liouville domain, respecting the orientation of W_φ.

Lemma 3.2.4 ([Gei08, p. 348])**.** *The contact structure constructed in Theorem 3.2.1 is supported by the open book decomposition induced by* $(W, \lambda; \psi)$.

Proof. Recall that the form defined in the theorem is defined by $\alpha = \lambda + d\varphi$ on $W_\eta(\psi)$ and by $fi^*\lambda + gd\varphi$ on $\partial W \times \mathbb{D}$. By restricting to the tangent space of a page, $d\varphi$ vanishes, and thus the pages are just $(W, d\lambda)$, which were assumed to be Liouville domains. □

3.2.1 Uniqueness of $OB(W, \lambda; \psi)$

It is of course natural to ask if the above construction of the contact manifold $OB(W, \lambda; \psi)$ is well-defined. We address this question to establish well-definedness up to contactomorphism, which also provides a strategy to prove when two contact manifolds are contactomorphic. We will utilizes this technique in the proof of Theorem 5.0.1. All these ideas are due to [Gir02].

Proposition 3.2.5. *Let M^{2n+1} be a closed oriented manifold and $\xi_i = \ker \alpha_i$ be two positive contact structures supported by the same open book decomposition (B, p) of M. Then ξ_0 and ξ_1 are isotopic.*

Proof. Let (W, λ_i) denote the abstract page of the open book decomposition, where we set $\lambda_i = \alpha_i|_{TW}$. Recall that $\partial \overline{W} = B$ and take a small tubular neighbourhood $B \times \mathbb{D}_\epsilon$. Let $h : [0, \epsilon] \to \mathbb{R}$ be a function with

- $h(0) = 0$, $h'(r) \geq 0$ near $r = 0$;
- $h \equiv 1$ for $r > \epsilon/2$.

Consider h as a function with domain $B \times \mathbb{D}_\epsilon$ and set for any $R > 0$

$$\alpha_{i,R} := \alpha_i + Rh(r)d\varphi.$$

A short computation shows that

$$\alpha_{i,R} \wedge (d\alpha_{i,R})^n = \alpha_i \wedge (d\alpha_i)^n + Rh(r)d\varphi \wedge (d\alpha_i)^n + Rh'(r)\alpha_i \wedge (d\alpha_i)^{n-1} \wedge dr \wedge d\varphi. \tag{3.2}$$

The first term is positive since the α_i are contact forms. The second term is nonnegative as $d\alpha_i$ is symplectic on the pages inducing the given orientation, which means by definition that any positive basis of the tangent space of a page together with ∂_φ is a positive basis of the tangent space of M. Hence, $d\varphi \wedge (d\alpha_i)^n$ is a positive volume form.

For the third term, let ϵ be small enough so that the intersection of the tubular neighbourhood $B \times \mathbb{D}_\epsilon$ with any page is contained within a symplectic collar of the pages. It does not matter whether we choose the Liouville vector field associated to α_1 or α_2 for the construction of the symplectic collar, as long as the flow preserves the orientation (which is clear as $\mathcal{L}_W(\omega) = \omega$ for any Liouville vector field).

Suppose for concreteness that we evaluate $\alpha_i \wedge (d\alpha_i)^{n-1} \wedge dr \wedge d\varphi$ at a point $x \in B \times \mathbb{D}_\epsilon \subset M$ such that $p(x) = \varphi_0$. By assumption on ϵ, we may identify

$$(B \times \mathbb{D}_\epsilon \cap W_{\varphi_0}, d\alpha_i|_{TW_{\varphi_0}}) \cong (B \times [-\epsilon, 0], d(e^t\lambda_i)).$$

Recall that as the α_i are Giroux forms, the λ_i are contact structures when restricted to B. Assume that $x = (p_0, t_0)$, which corresponds to $\phi_{t_0}(x)$ if ϕ

denotes the flow of the Liouville vector field used to construct the symplectic collar. Note that $t \in [-\epsilon, 0]$.

Now choose a positive basis $(u_1, \ldots, u_{2n-1})$ of $T_{p_0}B$ and transport it to $T_x W_{\varphi_0}$ by setting

$$v_j = D\phi_{t_0}(p_0)[u_j].$$

As ψ_t is an orientation-preserving diffeomorphism of W_{φ_0} and by the orientation conventions in Definition 3.2.2, $(\frac{\partial}{\partial t}, v_1, \ldots, v_{2n-1})$ is a positive basis of $T_x W_{\varphi_0}$ ($\frac{\partial}{\partial t}$ is outward-pointing on ∂W). Note that under the identification with the symplectic collar, we may express

$$v_j = u_j + t_0 \frac{\partial}{\partial t}|_{t_0}.$$

Our preliminary goal is to show that $\alpha_i \wedge (d\alpha_i)^{n-1} \wedge dr|_{TW_{\varphi_0}}$ is a volume form on $W_{\varphi_0} \cap B \times \mathbb{D}_\epsilon$. On the symplectic collar, we have

$$\begin{aligned}
\alpha_i \wedge (d\alpha_i)^{n-1} &= (e^t \lambda_i) \wedge (d(e^t \lambda_i))^{n-1} \\
&= e^t \lambda_i \wedge (e^t dt \wedge \lambda_i + e^t d\lambda_i)^{n-1} \\
&= e^t \lambda_i \wedge \Big(e^{(n-1)t} (d\lambda_i)^{n-1} + (n-1) e^{(n-1)t} dt \wedge \lambda_i \wedge (d\lambda_i)^{n-2}\Big) \\
&= e^{nt} \lambda_i \wedge (d\lambda_i)^{n-1}.
\end{aligned}$$

Evaluating $\alpha_i \wedge (d\alpha_i)^{n-1} \wedge dr|_{TW_{\varphi_0}}$ at the positive basis $(\frac{\partial}{\partial t}, v_1, \ldots, v_{2n-1})$ gives

$$-dr(\frac{\partial}{\partial t}) \cdot e^{nt_0} \lambda_i \wedge (d\lambda_i)^{n-1}(u_1, \ldots, u_{2n-1}).$$

The second factor is positive as the λ_i are positive contact forms on B and the u_j are a positive basis. Also, $dr(\frac{\partial}{\partial t}) < 0$ for ϵ small enough as r is decreasing along flow lines of the Liouville vector field in a neighbourhood of the binding, which proves that $\alpha_i \wedge (d\alpha_i)^{n-1} \wedge dr > 0$ in a neighbourhood of B (though not on B itself, as there, dr is not defined).

Finally, $d\varphi(\partial_\varphi) > 0$, so that

$$\alpha_i \wedge (d\alpha_i)^{n-1} \wedge dr \wedge d\varphi(v_1, \ldots, v_{2n-1}, \frac{\partial}{\partial t}, \partial_\varphi) > 0,$$

which establishes that also the third term is nonnegative. The last two terms in Equation (3.2) may vanish due to being multiplied by h or h', but the first is always positive. As we have shown that all terms evaluate to something nonnegative or positive on positively oriented bases, this establishes that both $\alpha_{i,R}$ are positive contact forms inducing the same contact structure as α_i.

We now obtain an isotopy between ξ_0 and ξ_1 by the convex combination

$$\alpha_t = (1-t)\alpha_{0,R} + t\alpha_{1,R},$$

which is contact for all $t \in [0,1]$ if R is large enough. □

As promised, we obtain that the contact structure on $OB(W, \lambda; \psi)$ does not depend on the choices made in the construction:

Corollary 3.2.6. *The contact manifold $OB(W, \lambda; \psi)$ is well-defined up to contactomorphism.*

Proof. The base manifold is always diffeomorphic to $OB(W; \psi)$, and if ξ_0 and ξ_1 are two contact structures on $OB(W; \psi)$ arising through different choices regarding f and g in the construction in Theorem 3.2.1, then by Lemma 3.2.4, both are supported by the open book decomposition induced by the abstract open book $(W, \lambda; \psi)$ on $OB(W; \psi)$. Thus we conclude immediately by the preceding proposition. □

Perhaps more interestingly, this provides a technique to prove that two contact manifolds are contactomorphic.

Corollary 3.2.7. *If two contact manifolds (M_0, ξ_0) and (M_1, ξ_1) admit supporting open book decompositions so that their respective abstract pages are symplectomorphic to the Liouville domain (W, λ) and their monodromies are symplectically isotopic to ψ, then (M_0, ξ_0) and (M_1, ξ_1) are contactomorphic.*

Proof. From the assumptions and the uniqueness proposition, it follows immediately that

$$(M_0, \xi_0) \cong OB(W, \lambda; \psi) \cong (M_1, \xi_1),$$

where $\cong$ denotes contactomorphism. □

Remark 3.2.8. In dimension 3, the interplay between open book decompositions of contact manifolds and their contact structures can be made more precise, which is the content of the celebrated Giroux Correspondence Theorem [Gir02]:

Theorem 3.2.9 (Giroux)**.** *Let M be a closed oriented 3-manifold. Then there is a one-to-one correspondence between*

$$\{\textit{oriented contact structures on } M \textit{ up to isotopy}\}$$

and

$$\{\textit{open book decompositions of } M \textit{ up to positive stabilization}\}.$$

The equivalence relation of positive stabilization of open books is defined by adding a 1-handle to the page and composing the monodromy with a right-handed Dehn twist along a closed simple embedded curve going exactly once around the handle.

3.2.2 Examples

Returning to the examples of S^3 we encountered earlier, we check if the standard contact structure on S^3 defined by

$$\alpha = r_1^2 d\varphi_1 + r_2^2 d\varphi_2$$

is supported by the open book decompositions considered in Section 3.1.2.

Example 3.2.10.

1. We first treated $B = \{(0, z_2) \in S^3\}$ together with $p(z_1, z_2) = \frac{z_1}{|z_1|}$ and found the corresponding abstract open book to be $(\mathbb{D}^2, \psi = \mathrm{id})$.

   On B, α restricts to $d\varphi_2$, which is the standard contact form on S^1 again. On the pages, $d\alpha = r_1 dr_1 \wedge d\varphi_1 + r_2 dr_2 \wedge d\varphi_2$ restricts to $r_2 dr_2 \wedge d\varphi_2$, as φ_1 is constant on the pages. This is a symplectic form, which shows that the the standard contact structure $\ker \alpha$ is supported by the open book decomposition (B, p).

2. For the other binding

   $$B' = \{(z_1, 0) \in S^3\} \cup \{(0, z_2) \in S^3\},$$

   we see that on $K_i \subset B$, the standard contact form α restricts to $d\varphi_i$, once more the standard contact form on S^1. Using the parametrisation of the pages

   $$(p')^{-1}(\varphi) = \left\{ (r_1 e^{i\varphi_1}, \sqrt{1 - r_1^2} e^{i(\varphi - \varphi_1)}) \right\}$$

   for $(r_1, \varphi_1) \in (0, 1) \times S^1$, we get

   $$d\alpha = r_1 dr_1 \wedge d\varphi_1 + \sqrt{1 - r_1^2} d(\sqrt{1 - r_1^2}) \wedge d(\varphi - \varphi_1) = 2r_1 dr_1 \wedge d\varphi_1.$$

   This is symplectic and hence (p', B') also supports $\ker \alpha$.

3.3 Symplectic Fillings by Lefschetz-Bott Fibrations

This section explains how to obtain symplectic fillings from Lefschetz-Bott fibrations using open books. We use this technique in Section 5.5 to exhibit different symplectic fillings of the A_k-type singularity.

Recall from the beginning of this chapter that a topological Lefschetz-Bott fibration (E, π, J, j) with regular fiber F over $\mathbb{D}$ admits a topological description of its total space as

$$\partial E = F(\psi) \cup_{\partial F \times S^1} (\partial F \times \mathbb{D}).$$

This suggests an identification of ∂E with the abstract open book (F, ψ), though one has to be careful about the fact E has a corner. On the other hand, the manifold $OB(F; \psi)$ is constructed with a gluing as discussed after Theorem 3.2.1, so that $OB(F; \psi)$ is closed.

Suppose the corner can be smoothed. We may then identify the resulting manifold E' with $OB(F; \psi)$ induced from the abstract open book (F, ψ). We have seen in Theorem 3.2.1 that $OB(F; \psi)$ admits a contact structure if we further endow the page F with a 1-form λ making $(F, d\lambda)$ into a Liouville domain; recall that we denoted the resulting contact manifold by $OB(F, \lambda; \psi)$.

It is then natural to attempt to expand this to the symplectic setting and ask if there is a notion of compatibility between the symplectic structure of the Lefschetz-Bott fibration and the contact structure on the boundary. In particular, one may be interested in the possibility that (E, Ω) could, after smoothing corners, serve as a strong symplectic filling of a contact open book associated to ∂E.

A strong symplectic filling $(W, \omega = d\lambda)$ of a contact manifold (M, α) must satisfy that ω is symplectic (evidently), and that ω is exact near the boundary with outward pointing Liouville vector field V. For a symplectic Lefschetz-Bott fibration (E, π, Ω) over $\mathbb{D}$, let us hence require that

(1) Ω is symplectic on all of E, and

(2) $\Omega = d\lambda$ is exact near ∂E, and both $\lambda|_{\partial_h E}$ and $\lambda|_{\partial_v E}$ are positive contact forms.

It turns out that one can "interpolate between" the Liouville vector fields associated to $\lambda|_{\partial_h E}$ and $\lambda|_{\partial_v E}$ (which are outward pointing along $\partial_h E$ and $\partial_v E$ as they are assumed to be positively contact) to obtain another Liouville vector field V which is transverse to the boundary of a manifold E' obtained by smoothing the corners of E (cf. [LHW18, Section 2.5]).

Thus if (1) and (2) hold, (E', Ω) is a strong symplectic filling of $(\partial E', \ker \lambda)$.

To obtain a contact open book description of $\partial E'$, we require that $(F; \psi)$ can be made into a Liouville domain. This suggests that we should impose

(3) $(\pi^{-1}(z) \setminus E^{\mathrm{crit}}, \Omega|_{\pi^{-1}(z)})$ is a Liouville domain for all $z \in \mathbb{D}$.

Denote a regular reference fiber by $(F, d\lambda) = (\pi^{-1}(1), \Omega|_{\pi^{-1}(1)})$. If (E, π, Ω) satisfies (1)-(3), Theorem 3.2.1 together with Lemma 3.2.4 give that λ is a Giroux form for the open book decomposition of the contact manifold $OB(F, \lambda; \psi)$.

Therefore, to establish that a symplectic Lefschetz-Bott fibration (E, π, Ω) as above induces a strong symplectic filling of the contact manifold $OB(F, \lambda; \psi)$, one needs to verify that

$$(E', \ker \lambda) \text{ is contactomorphic to } OB(F, \lambda; \psi).$$

This is shown in [Oba20, Proposition B.3], which also reviews the smoothing of the corners of E.

Let us state this as a proposition.

Proposition 3.3.1 ([LHW18, Section 2.5], [Oba20, Proposition B.3])**.** *Let (E, π, Ω) be a symplectic Lefschetz-Bott fibration over $\mathbb{D}$ with monodromy ψ and generic fiber the Liouville domain $(F, d\lambda) = (\pi^{-1}(z) \setminus E^{\mathrm{crit}}, \Omega|_{\pi^{-1}(z)})$. Suppose Ω is nondegenerate on E and exact near ∂E and on each regular fiber of π.*

Then Ω can be deformed and the corners of E can be smoothed so that (E, Ω) is a strong symplectic filling of the contact manifold $OB(F, \lambda; \psi)$

This proposition allows us in certain cases to read off fillability of a contact manifold.

Corollary 3.3.2 ([Oba20, Corollary B.4])**.** *Suppose a contact manifold (M, ξ) is supported by an open book decomposition with pages symplectomorphic to the Liouville domain $(V, \omega = d\lambda)$ and monodromy $\psi \in \mathrm{Symp}(V, \omega)$. Suppose that ψ is symplectically isotopic to the composition of right-handed fibered Dehn twists*

$$\psi \cong \tau_{C_1} \circ \ldots \circ \tau_{C_k}$$

for $C_1, \ldots, C_k$ a collection of spherically fibered coisotropic submanifolds. Then (M, ξ) is strongly fillable.

Remark 3.3.3. Recall that the total space of a Lefschetz fibration $\pi : E^{2n} \to \mathbb{D}$ admits a handlebody decomposition with no handles of index greater than n. If E satisfies (1), then in particular, there exists a global symplectic form $\Omega \in \Omega^2(E)$. Therefore, one can choose a compatible almost complex structure J on E and we are in position to apply Theorem 1.4.22 to conclude that there exists a Stein structure on E. In particular, any strong symplectic filling induced by a Lefschetz fibration is in fact a Stein filling.

Remark 3.3.4. [LHW18] is an extensive reference for results of this type for Lefschetz fibrations on 4-manifolds, see in particular theorem 1.24. It is shown that for topological Lefschetz fibrations (E, π), the space of symplectic forms $\Omega \in \Omega^2(E)$ making (E, π, Ω) into a symplectic Lefschetz fibration satisfying (1)-(3) are nonempty and contractible, and that picking any such form, the corners of E can be smoothed so that (E, Ω) is a strong symplectic filling of a contact manifold supported by the induced open book decomposition of the boundary.

If moreover the Lefschetz fibration is **allowable**, which means that none of its vanishing cycles is homologically trivial in the fiber, then the same is true for the following spaces:

- the space of symplectic forms on E giving π the structure of an exact symplectic Lefschetz fibration, thereby inducing a Liouville filling;

- the space of **almost Stein structures** (J, ϕ), which consist of an almost complex structure J and a J-convex function ϕ so that $-dd^{\mathbb{C}}\phi$ restricts to a contact form on the faces of ∂E. It is explained how this gives rise to a veritable Stein structure on E after smoothing corners, producing a Stein filling.

Conversely, Giroux and Pardon proved in [GP17] that every Stein domain W admits a Lefschetz fibration (W', π) whose fibers are Stein domains and so that W' can be deformed to W. This proves that any Stein domain admits a Lefschetz fibration, and that the total space of any Lefschetz fibration can be deformed to a Stein domain.

Remark 3.3.5. Note in particular that this result implies that any manifold which arises as the boundary of the total space of a Lefschetz fibration (with smoothed corners) admits a Stein fillable contact structure.

Chapter 4

Lefschetz-Bott Fibrations on Line Bundles

The aim of this chapter is to prove the following:

Theorem 4.0.1 ([Oba20, Theorem 1.1]). *Let (M, ω) be a closed symplectic manifold. Suppose that $[\omega/2\pi] \in H^2(M; \mathbb{R})$ has an integral lift Poincaré dual to the homology class of a symplectic hypersurface H in (M, ω). Then there exists a complex line bundle L over (M, ω) with first Chern class $c_1(L) = -[\omega/2\pi]$ which admits a symplectic Lefschetz-Bott fibration over $\mathbb{C}$ with fibers $M \setminus H$ and critical set H.*

Let us first recall the notion of Chern classes. Suppose $\pi : L \to M$ is a complex vector bundle of rank k. Then the r-th Chern class $c_r(L)$ is a cohomology class in $H^{2r}(M; \mathbb{Z})$. Their significance in the case of complex line bundles lies in the fact that the first Chern class $c_1(L) \in H^2(M; \mathbb{Z})$ turns out to be a complete invariant: complex line bundles over a manifold M are classified up to isomorphism by $c_1(L)$ (see e.g. [Hus94, Theorem 3.4]).

We refer to the classical textbook [MS74] for the general theory of characteristic classes, in particular to Chapter 14 for the theory on Chern classes, and to Appendix C for their relation to Chern-Weil theory. For our purposes, we content ourselves by giving a way to define the first Chern class: suppose $p : L \to M$ is a complex line bundle over a manifold M. Choose a connection form $\alpha \in \Omega^1(L)$. The curvature form associated to α is a 2-form $\beta \in \Omega^2(M)$ on the base satisfying $p^*\beta = d\alpha$. Then in our convention, the first Chern class of the bundle L is

$$c_1(L) = [-\beta/2\pi].$$

The assumptions on ω and H are motivated by an important result of Donaldson:

Theorem 4.0.2 ([Don96]). *Let (M, ω) be an integral closed symplectic manifold. Then there exists a sufficiently large integer $k > 0$ such that $[k\omega/2\pi]$ is Poincaré dual to the orientation class of a symplectic hypersurface H in M.*

The hypersurface H is called a **Donaldson hypersurface**, and in the following we normalize the symplectic form in order to assume $k = 1$.

Hypersurfaces of this type are also known as **symplectic divisors**, and a tuple (M, ω, H) consisting of an integral symplectic manifold (M, ω) together with a Donaldson hypersurface H is referred to as a **polarized manifold**. In [BC01], Biran and Cieliebak studied properties of polarized manifolds and provided numerous examples to which Theorem 4.0.1 could potentially be applied. One easy consequence from the definition is that polarized manifolds are exact away from the Donaldson hypersurface H.

Lemma 4.0.3. *Let (M, ω, H) be a polarized symplectic manifold. Then ω is exact away from H.*

Proof. Set $X = M \setminus H$ and denote the inclusion by $i : H \hookrightarrow M$. We show that $[i^*\omega/2\pi] = 0 \in H^2(X; \mathbb{R}) \cong \mathrm{Hom}(H_2(X; \mathbb{Z}); \mathbb{R})$, which by the de Rham isomorphism amounts to showing that for any 2-cycle $c \in C_2(X; \mathbb{Z})$, we have

$$\int_c i^*\omega/2\pi = 0.$$

Indeed, because $[\omega/2\pi]$ is Poincaré dual to $[H]$, we have

$$\int_c i^*\omega/2\pi = \int_{i(c)} \omega/2\pi = \int_{i(c)} PD[H] = i*[c] \cdot [H] = 0$$

since c and H are disjoint. □

The construction of the line bundle L and the Lefschetz-Bott fibration associated to a Donaldson hypersurface $H \subset (M, \omega)$ will proceed along the following program:

Outline

1. Building a local model: construct an associated S^1-bundle over H with a symplectic form (Section 4.1).
2. Construct a neighbourhood $\nu(H)$ of H which can be symplectically identified with the previous associated bundle, so that $M = \nu(H) \cup V$, where V is the complement of $\nu(H)$ (Section 4.2).
3. Define complex line bundles over V and $\nu(H)$ as an associated bundle to a suitable S^1-action on $\mathbb{C}$ and endow their total spaces with symplectic forms (Section 4.3); symplectically glue them to a line bundle L over M (Section 4.4).

4. Define a tentative Lefschetz-Bott fibration $\pi : L \to \mathbb{C}$ whose critical submanifold is the zero section of H (Section 4.5).

5. Deform the symplectic structure on L such that the fibers carry a standard symplectic structure (Section 4.6).

6. Construct an almost complex structure J on L such that the symplectic structure is normally Kähler near the zero section of H in L (Section 4.7).

7. Show that all the data constructed above define a symplectic Lefschetz-Bott fibration (also Section 4.7).

4.1 The Local Model

We will model a neighbourhood of the Donaldson hypersurface $H \subset M$ on a bundle associated to a special principal S^1-bundle, which we now define.

Definition 4.1.1. Let (M, ω) be a closed symplectic manifold. The **Boothby-Wang bundle** $p : (P, \alpha) \to (M, \omega)$ over (M, ω) is a principal S^1-bundle with connection 1-form $\alpha \in \Omega^1(P; \mathfrak{s}^1)$ such that $d\alpha = p^*\omega$, and such that α is a contact form.

Remark 4.1.2. As $\mathfrak{s}^1 \cong \mathbb{R}$, we view α as an ordinary 1-form on P with values in $\mathbb{R}$.

Definition 4.1.3. Suppose ω is **integral**, then in our convention, $[\omega/2\pi] \in H^2(M; \mathbb{R})$ lies in the image of the map $H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{R})$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$.

We call a preimage of $[\omega/2\pi]$ in $H^2(M; \mathbb{Z})$ an **integral lift** of ω.

A symplectic manifold (M, ω) whose symplectic form ω is integral is referred to as an **integral symplectic manifold**.

Fixing an integral lift of an integral symplectic form, the Boothby-Wang bundle over (M, ω) exists and is unique up to isomorphism (see [BW58]).

Proposition 4.1.4 (Properties of the Boothby-Wang bundle)**.** *Let $p : (P, \alpha) \to (M, \omega)$ be the Boothby-Wang bundle over (M, ω) and write*

$$\xi_s(p) = \left.\frac{d}{dt}\right|_{t=s} \exp(ts) \cdot p, \quad p \in P,$$

for the infinitesimal generator associated to the S^1-action on P. Then $\xi_1 = R_\alpha$ is the Reeb vector field of α, and all its orbits are periodic.

Proof. Since α is a connection form, $\imath_{\xi_s}\alpha = s$ for all $s \in \mathfrak{s}^1$, so $\imath_{\xi_1}\alpha = 1$. Note that as the action is fiber-preserving, we have $Dp(x)[\xi_s(p)] = 0$ for any $s \in \mathfrak{s}^1$ and $x \in P$, so that

$$\imath_{\xi_1}(d\alpha) = \imath_{\xi_1}(p^*\omega) = 0.$$

The flow of ξ_1 is $\varphi_t(x) = \exp(t) \cdot x$, which is evidently 2π-periodic as $\exp(t)$ is the element $e^{it} \in S^1$. $\square$

The relevant associated bundles are obtained from two actions of S^1 on $\mathbb{C}$. Consider the group homomorphisms $\rho, \bar{\rho} : S^1 \to S^1$ given by

$$\rho(\theta) = e^{2\pi i\theta}, \qquad \bar{\rho}(\theta) = e^{-2\pi i\theta}.$$

These define S^1-actions on $\mathbb{C}$ via $\theta \cdot z := \rho(\theta)z$, and similarly for $\bar{\rho}$. As S^1 also acts on P, consider the right action on the product $P \times \mathbb{C}$ given in the standard way by

$$(p, z) \cdot \theta := (p \cdot \theta, \theta^{-1} \cdot z).$$

The associated bundle $P \times_\rho \mathbb{C}$ is thus $P \times \mathbb{C}$ divided by the action

$$(p, z) \cdot \theta = (p \cdot \theta, e^{-2\pi i\theta} z),$$

and $P \times_{\bar{\rho}} \mathbb{C}$ is $P \times \mathbb{C}$ divided by the action

$$(p, z) \cdot \theta = (p \cdot \theta, e^{2\pi i\theta} z).$$

Lemma 4.1.5 ([Oba20, Section 2.1])**.** *Define the forms $\omega'_\alpha, \omega'_{\overline{\alpha}} \in \Omega^2(P \times \mathbb{C})$ by*

$$\omega'_\alpha = p^*\omega + d(r^2 d\theta) + d(r^2\alpha) = d((1 + r^2)(\alpha + d\theta));$$
$$\omega'_{\overline{\alpha}} = p^*\omega + d(r^2 d\theta) - d(r^2\alpha) = d((1 - r^2)(\alpha - d\theta)).$$

The coordinates (r, θ) denote polar coordinates on $\mathbb{C}$. Their kernels are given by

$$\ker(\omega'_\alpha)_{(p,z)} = (R_\alpha - \frac{\partial}{\partial\theta})_{(p,z)}$$
$$\ker(\omega'_{\overline{\alpha}})_{(p,z)} = (R_\alpha + \frac{\partial}{\partial\theta})_{(p,z)}$$

for all $(p, z) \in P \times \mathbb{C}$, that is, it is spanned by the generator of the corresponding S^1-action. Hence ω'_α and $\omega'_{\overline{\alpha}}$ descend to symplectic forms

$$\omega_\alpha \in \Omega^2(P \times_\rho \mathbb{C}), \qquad \omega_{\overline{\alpha}} \in \Omega^2(P \times_{\bar{\rho}} \mathring{\mathbb{D}}).$$

Remark 4.1.6. We may define a zero section not only for vector bundles, but for any fiber bundle with groups as fibers by sending elements in the base to the identity element in their fiber.

Remark 4.1.7. Away from the zero section, $\omega_{\overline{\alpha}}$ is exact with primitive

$$\lambda_{\overline{\alpha}} = (1 - r^2)(\alpha - d\theta).$$

The local model for our neighbourhood of H in M will be the symplectic manifold

$$(P \times_{\bar{\rho}} \mathring{\mathbb{D}}, \omega_{\overline{\alpha}}),$$

where P is now the Boothby-Wang bundle over the Donaldson hypersurface H.

4.2 The Neighbourhood of H

Here we construct a neighbourhood of a symplectic hypersurface H modelled on the associated bundle $P_H \times_{\bar{\rho}} \mathring{\mathbb{D}}$ constructed above. Let (M, ω) be an integral symplectic manifold with a Donaldson hypersurface $H \subset M$, and denote by $i : H \hookrightarrow M$ the inclusion. Set $\omega_H := i^*\omega$ and denote by

$$p : (P, \alpha) \to (H, \omega_H)$$

the Boothby-Wang bundle over (H, ω_H). The construction in the previous subsection gives rise to a bundle

$$P \times_{\bar{\rho}} \mathring{\mathbb{D}} \to H$$

whose total space carries a symplectic form $\omega_{\bar{\alpha}}$ which is exact away from the zero section $i_0(H)$ with primitive $\lambda_{\bar{\alpha}}$. The following proposition allows us to think of a tubular neighbourhood of H as a convex neighbourhood of the zero section in $P \times_{\bar{\rho}} \mathring{D}$. Recall from Lemma 4.0.3 that ω is exact away from H.

Proposition 4.2.1 ([DL19, Lemma 2.2])**.** *Let H be a symplectic hypersurface of an integral symplectic manifold (M, ω) with $PD[H] = [\omega/2\pi]$. Then there exists some $\delta \in (0, 1)$, a primitive $\lambda \in \Omega^1(M \setminus H)$ of ω, and a symplectic embedding ν such that*

$$\begin{tikzcd}
(P \times_{\bar{\rho}} \mathbb{D}(\delta), \omega_{\bar{\alpha}}) \arrow[rr, hook, "\nu"] & & (M, \omega) \\
& H \arrow[ul, hook, "i_0"] \arrow[ur, hook', "i"'] &
\end{tikzcd}$$

commutes, and moreover $\nu^*\lambda = \lambda_{\bar{\alpha}}$.

The proof relies on the Weinstein tubular neighbourhood theorem combined with [DL19, Lemma 2.2] by Diogo and Lisi.

The desired neighbourhood is now given by

$$\nu_M(H) := \nu(P \times_{\bar{\rho}} \mathbb{D}(\delta)).$$

Also set

$$V := M \setminus \mathring{\nu}_M(H),$$

so that $M = \nu_M(H) \cup M$. Note that $V \cap \nu_M(H) = \partial\nu_M(H)$.

4.3 Line Bundles over the Decomposed Manifold

We build a complex line bundle over both V and $\nu_M(H)$ and endow their total spaces with a symplectic form. Let us begin with the bundle over V.

The Bundle over V

Denote by $p_V : V \times S^1 \to V$ the projection to V and let S^1 act on $V \times S^1$ by

$$(x, \theta_1) \cdot \theta := (x, \theta_1 + \theta),$$

where θ_1 is the coordinate on S^1. We may regard p_V as a principal S^1-bundle. Letting S^1 act on $\mathbb{C}$ via ρ, consider the associated bundle

$$(V \times S^1) \times_\rho \mathbb{C} \to V.$$

To construct a symplectic form on the total space, endow p_V with a connection form

$$\alpha_V = \lambda + d\theta_1.$$

This is indeed a connection form: the infinitesimal generator is given by $\xi_s = s\frac{\partial}{\partial\theta_1}$, and thus $\alpha_V(\xi_s) \equiv s$ for all $s \in \mathfrak{s}^1 \cong \mathbb{R}$. Moreover, α_V is evidently invariant under the action.

Let (r_2, θ_2) denote coordinates on $\mathbb{C}$ and define a form in $\Omega^2((V \times S^1) \times \mathbb{C})$ by

$$p_V^*(d\lambda) + d(r_2^2 d\theta_2) + d(r_2^2 \alpha_V) = d((1 + r_2^2)(\lambda + d\theta_1 + d\theta_2)).$$

This is S^1-invariant and hence descends to a symplectic form $\omega_{\alpha_V} \in \Omega^2((V \times S^1) \times_\rho \mathbb{C})$.

In fact, this bundle is symplectomorphic to the trivial bundle $\Pi_V : V \times \mathbb{C} \to V$ equipped with the symplectic form

$$\Omega_V = d\left((1 + r^2)(\lambda + d\theta)\right),$$

which is the bundle over V we are interested in. The symplectomorphism is given by

$$\begin{aligned} \Psi_V : (V \times S^1) \times_\rho \mathbb{C} &\to V \times \mathbb{C} \\ [x, \theta_1, (r_2, \theta_2)] &\mapsto (x, (r_2, \theta_1 + \theta_2)). \end{aligned}$$

The Bundle over $\nu_M(H)$

As for the bundle over $\nu_M(H) \cong P \times_{\bar\rho} \mathbb{D}(\delta)$, consider the quotient projection $p_\nu : P \times \mathbb{D}(\delta) \to P \times_{\bar\rho} \mathbb{D}(\delta)$. This is a principal S^1-bundle, and

$$\alpha_\nu = (1 - r_1^2)\alpha + r_1^2 d\theta_1$$

is well-defined and a connection form: the infinitesimal generator is $\xi_s' + s\frac{\partial}{\partial\theta_1}$ for ξ_s' the infinitesimal generator of the S^1-action on P, so that

$$\alpha_\nu(\xi_s' - s\frac{\partial}{\partial\theta_1}) = (1 - r_1^2)s + r_1^2 s = s.$$

Invariance under the action follows similarly from the fact that α and $d\theta$ are connection forms on P and S^1, respectively.

To obtain a bundle over $\nu_M(H)$, we compose with ν so that

$$\nu \circ p_\nu : P \times \mathbb{D}(\delta) \to \nu_M(H)$$

is a principal S^1-bundle with the same connection form. To make this into a complex line bundle, set

$$\begin{aligned} \Pi_\nu : (P \times \mathbb{D}(\delta)) \times_\rho \mathbb{C} &\to \nu_M(H) \\ [(x, (r_1, \theta_1), (r_2, \theta_2))] &\mapsto \nu([x, (r_1, \theta_1)]). \end{aligned}$$

The form on $P \times \mathbb{D}(\delta) \times \mathbb{C}$ defined by

$$d\alpha_\nu + d(r_2^2 d\theta_2) + d(r_2^2 \alpha_\nu) = d((1 + r_2^2)(\alpha_\nu + d\theta_2))$$

descends to a symplectic form $\omega_{\alpha_\nu} \in \Omega^2((P \times \mathbb{D}(\delta)) \times_\rho \mathbb{C})$.

4.4 Gluing to a Bundle over M

We now show that the bundles Π_V and Π_ν glue together symplectically to a bundle over $M = V \cup \nu_M(H)$. The first step is to slightly modify our definition of $\nu_M(H)$ and V in such a way that they overlap in an open set on which we can define a collar neighbourhood. On this collar, we will define a gluing symplectomorphism.

The modification of $\nu_M(H)$ consists in slightly shrinking δ and taking some $\delta' > \delta$ such that the tubular neighbourhood

$$\nu_M(H) = \nu(P \times_{\bar{\rho}} \mathbb{D}(\delta))$$

is contained in $\nu(P \times_{\bar{\rho}} \mathbb{D}(\delta'))$. Note that shrinking δ shrinks the tubular neighbourhood $\nu_M(H)$ and thus enlarges its complement V. We may view the overlap of $\nu(P \times_{\bar{\rho}} \mathbb{D}(\delta))$ and $\nu(P \times_{\bar{\rho}} \mathbb{D}(\delta'))$, which is

$$\nu_V(\partial V) := \nu(P \times_{\bar{\rho}} \mathbb{A}(\delta, \delta')) \hookrightarrow (M, \omega),$$

as a symplectically embedded annulus bundle, where $\mathbb{A}(\delta, \delta') = \{z \in \mathbb{C} \mid |z| \in [\delta, \delta']\}$. Note that the image does in fact lie in V, and that $\nu(P \times_{\bar{\rho}} S^1(\delta)) = \partial V$; hence we may view $\nu_V(\partial V)$ as our desired collar neighbourhood of ∂V in V.

Lemma 4.4.1. *The gluing map defined by*

$$\Phi : (P \times \mathbb{A}(\delta, \delta')) \times_\rho \mathbb{C} \to \nu_V(\partial V) \times \mathbb{C} \tag{4.1}$$

$$[x, (r_1, \theta_1), (r_2, \theta_2)] \mapsto (\nu([x, (r_1, \theta_1)]), (r_2, \theta_1 + \theta_2)). \tag{4.2}$$

is a symplectomorphism.

Proof. Recall that the symplectic form on $\nu_V(\partial V) \times \mathbb{C} \subset V \times \mathbb{C}$ is

$$\Omega_V = d\left((1+r^2)(\lambda + d\theta)\right),$$

and that $\omega_{\alpha_\nu} \in \Omega^2((P \times \mathbb{D}(\delta')) \times_\rho \mathbb{C})$ is the reduction of

$$d\left((1+r_2^2)(\alpha_\nu + d\theta_2)\right) \in \Omega^2(P \times \mathbb{D}(\delta') \times \mathbb{C}),$$

where $\alpha_\nu = (1-r_1^2)\alpha + r_1^2 d\theta_1$ and α is the connection-contact form on P.

Further recall from Proposition 4.2.1 that the embedding $\nu : P \times_{\bar{\rho}} \mathbb{D}(\delta') \hookrightarrow (M, \omega)$ satisfies $\nu^*\lambda = (1-r^2)(\alpha - d\theta)$. Combining this with

$$\Phi^* r = r_2, \qquad \Phi^*\theta = \theta_1 + \theta_2,$$

a direct computation shows that

$$\begin{aligned}
\Phi^*\left((1+r^2)(\lambda + d\theta)\right) &= \left((1+\Phi^* r^2)(\Phi^*\lambda + d\Phi^*\theta)\right) \\
&= \left((1+r_2^2)((1-r_1^2)(\alpha - d\theta_1) + d\theta_1 + d\theta_2)\right) \\
&= \left((1+r_2^2)((1-r_1^2)\alpha + r_1^2 d\theta_1 - d\theta_1 + d\theta_1 + d\theta_2)\right) \\
&= \left((1+r_2^2)(\alpha_\nu + d\theta_2)\right).
\end{aligned}$$

As the domain of Φ does not contain (the zero section of) H, the symplectic forms are exact on the entire domain and target, so this argument suffices to establish that Φ is a symplectomorphism $\square$

We may hence consider the glued bundle

$$\Pi : V \times \mathbb{C} \cup_\Phi (P \times \mathbb{D}(\delta')) \times_\rho \mathbb{C} \to V \cup \nu_M(H) = M,$$

defined by

$$\Pi(x, (r, \theta)) = \Pi_V(x, (r, \theta)) = x$$

on $V \times \mathbb{C}$, and by

$$\Pi([x, (r_1, \theta_1), (r_2, \theta_2)]) = \Pi_\nu([x, (r_1, \theta_1), (r_2, \theta_2)]) = \nu([x, (r_1, \theta_1)])$$

on $(P \times \mathbb{D}(\delta')) \times_\rho \mathbb{C}$. Denote this bundle by $\Pi : L \to M$.

Let us determine the Chern class $c_1(L)$. Note that due to the correspondence of rank-k complex vector bundles and principal $U(k)$-bundles, the Chern class of an associated bundle $P \times_\rho \mathbb{C}$ is given by the Chern class of the associated bundle P.

We will thus show that the curvature forms associated to the connection forms chosen on the bundles over V and $\nu_M(H)$ are given by $\omega|_V$ and $\omega|_\nu$, respectively, which implies that the Chern class is $c_1(L) = [-\omega/2\pi]$.

Recall that the bundle over V was defined as $(V \times S^1) \times_\rho \mathbb{C}$, where we endowed $V \times S^1$ with the connection form $\alpha_V = \lambda + d\theta_1$. Then we have

$$d\alpha_V = d\lambda = p_V^*\omega,$$

so that $\omega|_V$ is indeed the curvature form of the bundle over V.

As for the bundle over $\nu_M(H)$, the connection form on $\nu \circ p_\nu : P \times \mathbb{D}(\delta') \to \nu_M(H)$ was chosen to be

$$\alpha_\nu = (1 - r_1^2)\alpha + r_1^2 d\theta_1.$$

Then $d\alpha_\nu = d((1 - r_1^2)(\alpha - d\theta_1))$, which coincides with $p_\nu^* \omega_{\bar{\alpha}}$. On the other hand, we have

$$\nu^* \omega = \omega_{\bar{\alpha}}$$

by Proposition 4.2.1, so that $(\nu \circ p_\nu)^* \omega = d\alpha\nu$, establishing that the curvature form of α_ν is $\omega|_{\nu_M(H)}$.

4.5 A Fibration on the Line Bundle

Here we define what will be a Lefschetz-Bott fibration on the space L we constructed above. Following the notation in [Oba20], abbreviate $P(\mathbb{D}(\delta'), \mathbb{C}) := (P \times \mathbb{D}(\delta')) \times_\rho \mathbb{C}$. We define the fibration separately over each piece of $L = V \times \mathbb{C} \cup_\Phi P(\mathbb{D}(\delta'), \mathbb{C})$ as follows:

$$\pi_V : V \times \mathbb{C} \to \mathbb{C}, \qquad (x, (r, \theta)) \mapsto (r, \theta),$$

and

$$\pi_\nu : P(\mathbb{D}(\delta'), \mathbb{C}) \to \mathbb{C}, \qquad [x, (r_1, \theta_1), (r_2, \theta_2)] \mapsto (\mu(r_1) r_2, \theta_1 + \theta_2).$$

Here, $\mu : \mathbb{R} \to \mathbb{R}$ is a smooth function with

- $\mu(r) = r$ for $r \leq \epsilon$, where $0 < \epsilon < \delta$;
- $\mu(r) \equiv 1$ for $r \geq \delta$;
- $\mu'(r) \geq 0$ for all r.

Lemma 4.5.1. *The map $\pi := \pi_V \cup \pi_\nu : M \to \mathbb{C}$ defined by π_V on V and π_ν on $\nu_M(H)$ is well-defined.*

Proof. We first show that the map π_ν is well-defined, and then that π respects the gluing by Φ.

The map π_ν is well-defined since S^1 acts on $\mathbb{D}(\delta')$ via $\bar{\rho}$ and on $\mathbb{C}$ via ρ:

$$\begin{aligned}
\pi_\nu([(x, (r_1, \theta_1), (r_2, \theta_2)) \cdot \theta]) &= \pi_\nu([x \cdot_P \theta, (r_1, \theta_1) \cdot_{\bar{\rho}} \theta^{-1}, (r_2, \theta_2) \cdot_\rho \theta^{-1}]) \\
&= \pi_\nu([x \cdot_P \theta, (r_1, \theta_1 + \theta), (r_2, \theta_2 - \theta)]) \\
&= (\mu(r_1) r_2, \theta_1 + \theta_2) \\
&= \pi_\nu([x, (r_1, \theta_1), (r_2, \theta_2)]).
\end{aligned}$$

To show that $\pi := \pi_V \cup \pi_\nu : L \to \mathbb{C}$ is well-defined, we need to verify that for $[(x, (r_1, \theta_1), (r_2, \theta_2))] \in (P \times \mathbb{A}(\delta, \delta')) \times_\rho \mathbb{C}$, we have

$$\pi_\nu \left([(x, (r_1, \theta_1), (r_2, \theta_2))]\right) \stackrel{!}{=} \pi_V \left(\Phi([(x, (r_1, \theta_1), (r_2, \theta_2))])\right).$$

So we compute the right hand side, which is

$$\begin{aligned}\pi_V \left(\Phi([(x, (r_1, \theta_1), (r_2, \theta_2))])\right) &= \pi_V(\nu(x, (r_1, \theta_1)), (r_2, \theta_1 + \theta_2)) \\ &= (r_2, \theta_1 + \theta_2).\end{aligned}$$

Since $(r_1, \theta_1) \in \mathbb{A}(\delta, \delta')$, we have $\mu(r_1) = 1$, which means π is well-defined. $\square$

Note that we may write π on $P(\mathbb{D}(\delta'), \mathbb{C})$ as

$$\pi([x, z_1, z_2]) = \frac{\mu(|z_1|)}{|z_1|} z_1 z_2, \tag{4.3}$$

and that near H_0, specifically for $|z_1| \leq \epsilon$, we have

$$\pi([x, z_1, z_2]) = z_1 z_2.$$

Proposition 4.5.2 ([Oba20, Section 3.3])**.** *The critical point set of π is*

$$H_0 = \{[x, z_1, z_2] \in P(\mathbb{D}(\delta'), \mathbb{C}) \mid z_1 = z_2 = 0\}.$$

Proof. We have

$$d\pi_{[x,z_1,z_2]} = \mu'(r_1) r_2 dr_1 + \mu(r_1) dr_2 + d\theta_1 + d\theta_2$$

on $P(\mathbb{D}(\delta'), \mathbb{C})$, which vanishes precisely for $r_1 = r_2 = 0$. On $V \times \mathbb{C}$, there are no critical points. $\square$

Remark 4.5.3. The critical manifold H_0 can be seen as the zero section of the bundle

$$P(\mathbb{D}(\delta'), \mathbb{C}) \to H;$$

Indeed, the zero section of $P \times \mathbb{D}(\delta') \to H$ is just $\{(1_y, 0) \in P \times \mathbb{D}(\delta') \mid y \in H\}$, where 1_y denotes the neutral element of the fiber $P_y \cong S^1$. By definition of principal bundles, the fibers of P are precisely the orbits of the action by S^1, so that $(1_y, 0)$ and $(x, 0)$ are in the same orbit for all $x \in P_y$, which means that when passing to the associated bundle, the zero section is H_0, as claimed.

Note that $\pi(H_0) = 0 \in \mathbb{C}$, so that 0 is the only critical value of π.

4.6 Deformation of the Symplectic Structure

Recall once more that the symplectic structure $\omega_{\alpha_\nu} \in \Omega^2(P(\mathbb{D}(\delta), \mathbb{C}))$ is the reduction of

$$d\alpha_\nu + d(r_2^2 d\theta_2) + d(r_2^2 \alpha_\nu) = d\left((1 + r_1^2)(\alpha_\nu + d\theta_2)\right) \in \Omega^2(P \times \mathbb{D}(\delta') \times \mathbb{C}),$$

where $\alpha_\nu = (1 - r_1^2)\alpha + r_1^2 d\theta_1$. For ϵ as in the definition of the function μ we used to define the fibration π before Lemma 4.5.1, define a neighbourhood of the critical locus

$$\nu_\epsilon(H_0) := \{[x, z_1, z_2] \in P(\mathbb{D}(\delta'), \mathbb{C}) \mid |z_1|^2 + |z_2|^2 \leq \epsilon\} \subset P(\mathbb{D}(\delta'), \mathbb{C}).$$

Recall that for a symplectic Lefschetz-Bott fibration, we require the fibers to be symplectic, and for a neighbourhood of the critical locus to be foliated by slices on which there is a complex structure which is Kähler for the symplectic structure. This is the neighbourhood on which we are going to exhibit such a complex structure.

Considering the restricted bundle $\nu_\epsilon(H_0) \to H$, the fibers may be regarded as $\mathbb{D}(\delta') \times \mathbb{D}(\epsilon)$, on which the standard symplectic structure is

$$\omega_0 = r_1 dr_1 \wedge d\theta_1 + r_2 dr_2 \wedge d\theta_2.$$

Observe, however, that ω_{α_ν} restricted to the fibers lifts to a different structure. We now deform the symplectic structure near the critical locus to attain the standard symplectic form on the fibers.

Lemma 4.6.1 ([Oba20, Lemma 3.5])**.** *There exists a symplectic form $\Omega_\nu \in \Omega^2(P(\mathbb{D}(\delta'), \mathbb{C}))$ with the following properties:*

- *Ω_ν coincides with ω_{α_ν} outside of $\nu_\eta(H_0)$ for some $\epsilon < \eta < 1$;*
- *Ω_ν restricts to twice the standard symplectic form on the fibers in $\nu_\epsilon(H_0)$.*

Moreover, there exists a symplectomorphism between $(P(\mathbb{D}(\delta'), \mathbb{C}), \Omega_\nu)$ and $(P(\mathbb{D}(\delta'), \mathbb{C}), \omega_{\alpha_\nu})$ supported in $\nu_\eta(H_0)$.

Proof. **Construction of Ω_ν**

To construct Ω_ν, pick a smooth function $u : \mathbb{R} \to \mathbb{R}$ and some $0 \leq \epsilon', \epsilon''$ with $\epsilon < \epsilon' < \epsilon'' < 1$ so that

- $u(s) \equiv 0$ for $s \leq \epsilon'$;
- $u(s) \equiv 1$ for $s \geq \epsilon''$;
- $u'(s) \geq 0$ for all s.

Now set $f(r_1, r_2) = u(r_1^2 + r_2^2)$ for $(r_1, r_2) \in [0, \delta') \times [0, \infty)$, and the 2-form

$$\tilde{\Omega}_\nu = d((1 + r_2^2)(d\theta_2 + \alpha)) + d((1 + f(r_1, r_2) r_2^2) r_1^2 (d\theta_1 - \alpha)).$$

The form is S^1-invariant and hence descends to a 2-form $\Omega_\nu \in \Omega^2(P(\mathbb{D}(\delta'), \mathbb{C}))$. Outside of $\nu_{\epsilon''}(H_0)$, we have $f(r_1^2 + r_2^2) \equiv 1$, so that in this region, we have

$$\tilde{\Omega}_\nu = d((1 + r_2^2)(d\theta_2 + \alpha)) + d((1 + r_1^2 r_2^2)(d\theta_1 - \alpha)).$$

A quick computation shows that this is agrees with ω_{α_ν}.

Nondegeneracy of Ω_ν

We now prove that Ω_ν is a symplectic form and find a symplectomorphism

$$(P(\mathbb{D}(\delta'), \mathbb{C}), \omega_{\alpha_\nu}) \to (P(\mathbb{D}(\delta 0), \mathbb{C}), \Omega_\nu).$$

For nondegeneracy, note first that $\dim(M) = 2n$, $\dim(H) = 2n - 2$, and hence $\dim(P) = 2n - 1$. Since of course $\dim(S^1) = 1$, this implies $\dim(P(\mathbb{D}(\delta'), \mathbb{C})) = 2n + 2$. One can now compute

$$\tilde{\Omega}_\nu^{n+1} = C(r_1, r_2) dr_1^2 \wedge (d\theta_1 - \alpha) \wedge dr_2^2 \wedge (d\theta + \alpha) \wedge (d\alpha)^{n-1}$$

for $C(r_1, r_2) = n(n+1)(1 - r_1^2 + r_2^2(1 - f r_1^2))^{n-1}(r_1^2 r_2^2 u' + f r_2^2 + 1)$. This function is strictly positive, and the wedge product of forms following it is a volume form (recall that α is contact).

We will obtain a symplectomorphism by applying the Moser trick. We have that

$$\omega_{\alpha_\nu} - \Omega_\nu = d(r_1^2 r_2^2 (d\theta_1 - \alpha)(1 - f(r_1, r_2))),$$

which is exact. Thus ω_{α_ν} and Ω_ν are cohomologous, as well as the convex combinations $\omega_t := (1 - t)\omega_{\alpha_\nu} + t\Omega_\nu$ for $t \in [0, 1]$. Therefore, by Moser's argument, there exists an isotopy φ_t of $P(\mathbb{D}(\delta'), \mathbb{C})$ with $\varphi_t^* \omega_t = \omega_0 = \omega_{\alpha_\nu}$.

Let us repeat Moser's argument in this case to show that the symplectomorphism is supported in $\nu_{\epsilon''}(H_0)$. Since the ω_t are cohomologous, we may find a smoothly varying family of 1-forms β_t so that $\dot{\omega}_t = -d\beta_t$. Recall that the isotopy φ_t is the flow of the time-dependent vector field V_t defined by

$$\imath_{V_t} \omega_t = \beta_t.$$

This flow satisfies

$$\frac{d}{dt} \varphi_t^* \omega_t = \varphi_t^* \left(\mathcal{L}_{V_t} \omega_t + \dot{\omega}_t\right) = \varphi_t^* \left(d \imath_{V_t} \omega_t - d\beta_t\right) = 0,$$

which proves that φ_t is the desired isotopy.

Outside of $\nu_{\epsilon''}(H_0)$, we have remarked that $\omega_{\alpha_\nu} - \Omega_\nu = 0$, so that also $\beta_t = 0$. Thus we have $\imath_{V_t} \omega_t = 0$, which by nondegeneracy implies $V_t = 0$ outside of $\nu_{\epsilon''}(H_0)$. The flow φ_t is hence the identity in this region, proving the claim.

Restriction to the fibers

The last claim to verify is that Ω_ν restricts to twice the standard symplectic form on the fibers of the bundle $\nu_\epsilon(H_0) \to H$. Let j_x denote the inclusion of the fiber over $x \in H$. As $\epsilon \leq \epsilon'$, by choice of u, the restricted form is

$$\begin{aligned} j_x^*\Omega_\nu &= d((1+r_2^2)d\theta_2) + d(r_1^2 d\theta_1) \\ &= 2r_2 dr_2 \wedge d\theta_2 + 2r_1 dr_1 \wedge d\theta_1, \end{aligned}$$

as claimed. □

4.7 Construction of the Kähler Structure

Our setting is now that of a line bundle L given by

$$\Pi : V \times \mathbb{C} \cup_\Phi P(\mathbb{D}(\delta'), \mathbb{C}) \to M,$$

which is the trivial projection to V on $V\times\mathbb{C}$, and which maps $[x,(r_1,\theta_1),(r_2,\theta_2)] \in P(\mathbb{D}(\delta'),\mathbb{C})$ to $\nu(x,(r_1,\theta_1)) \in \nu_M(H) \subset M$. $V \times \mathbb{C}$ carries the symplectic form Ω_V, and we endow $P(\mathbb{D}(\delta'),\mathbb{C})$ with Ω_ν constructed in Section 4.6. As Ω_ν coincides with ω_{α_ν} outside of $\nu_\epsilon(H_0)$, the symplectic gluing via Φ goes through without change so that L carries a global symplectic form Ω defined on the two factors by

$$\Omega|_{V\times\mathbb{C}} = \Omega_V \text{and } \Omega|_{P(\mathbb{D}(\delta'),\mathbb{C})} = \Omega_\nu.$$

Remark 4.7.1. The definition of a symplectic Lefschetz-Bott fibration only requires the 2-form on the total space to be symplectic on the vertical subbbundle, not globally. The 2-form $\Omega \in \Omega^2(L)$ constructed here is globally symplectic, however.

We are now ready to construct the final piece of data constituting a Lefschetz-Bott fibration.

Lemma 4.7.2 ([Oba20, Lemma 3.6])**.** *There exists an almost complex structure J on $P(\mathbb{D}(\delta'),\mathbb{C})$ compatible with Ω_ν such that H_0 is an almost complex submanifold of $(\nu_\epsilon(H_0), J)$, and Ω_ν is normally Kähler near H_0.*

Proof. We first construct a Riemannian metric on each factor of $P \times \mathbb{D}(\delta') \times \mathbb{C}$ which is invariant under the action of S^1, so that it descends to $P(\mathbb{D}(\delta'),\mathbb{C})$.

Further recall that α is a contact form on P, so that $d\alpha$ is a symplectic form on $\ker\alpha \subset TP$. Choose an almost complex structure J_α which is compatible with $d\alpha$, which means $d\alpha(\cdot, J_\alpha \cdot)$ is a Riemannian metric on $\ker\alpha$. To make this into a Riemannian metric on all of TP, set

$$g_\alpha(u,v) := d\alpha(u, J_\alpha v) + \alpha(u)\alpha(v).$$

The forms α and $d\alpha$ are S^1-invariant by virtue of α being a connection form, so that also g_α is invariant under the action of S^1 on P. Endow $\mathbb{D}(\delta')$ and $\mathbb{C}$ with the standard Riemannian metrics g_1 and g_2, which are also S^1-invariant, as the action is just a rotation. Hence,

$$g := g_\alpha + g_1 + g_2$$

induces a Riemannian metric on the quotient $P(\mathbb{D}(\delta), \mathbb{C})$. Thus the polar decomposition of $(P(\mathbb{D}(\delta'), \mathbb{C}), \Omega_\nu)$ with respect to g (see [Can06, Chapter 12]) provides a compatible almost complex structure J on $P(\mathbb{D}(\delta'), \mathbb{C})$ (though the induced metric need not necessarily coincide with g). Restricting to H_0, J is the almost complex structure arising from the polar decomposition of g_α (on the quotient $P(\mathbb{D}(\delta'), \mathbb{C})$), and thus H_0 is an almost complex submanifold of $(P(\mathbb{D}(\delta'), \mathbb{C}), J)$.

With respect to J, Ω_ν is normally Kähler near H_0: we can foliate the tubular neighbourhood $\nu_M(H_0)$ of the critical locus by normal slices D_x for $[x, 0, 0]$ ranging in H_0, where

$$D_x = \{[x, z_1, z_2] \in P(\mathbb{D}(\delta'), \mathbb{C}) \mid |z_1|^2 + |z_2|^2 < \epsilon\}.$$

The almost complex structure J restricted to D_x is just the standard complex structure on $\mathbb{C}^2$, and $\Omega_\nu|_{D_x} = 2(r_1 dr_1 \wedge d\theta_1 + r_2 dr_2 \wedge d\theta_2)$ is twice the standard symplectic structure. In particular, $J|_{D_x}$ is integrable and compatible with $\Omega_\nu|_{D_x}$, which is closed, and hence Kähler. $\square$

With all this in place, it is now straightforward to prove

Theorem 4.7.3 ([Oba20, Section 3.3])**.** *The tuple (L, π, Ω, J, j_0), where j_0 is the standard complex structure on $\mathbb{C}$, is a symplectic Lefschetz-Bott fibration.*

Proof. The critical locus of π is H_0, which is a smooth submanifold of L with finitely many connected components. Near H_0, more precisely, on $\nu_\epsilon(H_0)$, Ω is normally J-Kähler by Lemma 4.7.2. Three points remain to be shown:

- The form Ω is nondegenerate on $\ker D\pi$: note that $\ker D\pi_\nu = TP/S^1 \cong TH$, and that $\ker D\pi_V = TV$. Restricting Ω_V to TV is $d\lambda$, which is the symplectic form $\omega \in \Omega^2(M)$, whereas nondegeneracy of Ω_ν restricted to $1TP/S^1$ follows by a computation.

- The fibration π is (J, j_0)-holomorphic: on the normal slices D_x, $\mu(r_1) = r_1$, and thus $\pi|_{D_x}$ may be written as

  $$\pi|_{D_x}([x, z_1, z_2]) = z_1 z_2.$$

  As J is the standard complex structure on the $\mathbb{D}(\delta')$- and $\mathbb{C}$-factors, $\pi|_{D_x}$ is clearly (J, j_0)-holomorphic, and we see that J preserves TD_x and its orthogonal complement with respect to g, which consists of the tangent vectors in TP. Since $TP \subset \ker(D\pi)$, we have for any $u + v \in TD_x \oplus TD_x^\perp$ that

  $$D\pi[u + v] = D\pi[u].$$

Thus the fact that J preserves both TD_x and $TD_x^{\perp}$ yields immediately that

$$D\pi_{[x,z_1,z_2]}[J(u+v)] = D\pi_{[x,z_1,z_2]}[Ju] = j_0 D\pi_{[x,z_1,z_2]}[J(u+v)],$$

so that π is holomorphic on all of $\nu_\epsilon(H)$.

- The complex normal Hessian is everywhere nondegenerate: the local expression for π on the normal slices D_x yields immediately that at any $[x, z_1, z_2] \in D_x$, the holomorphic normal Hessian is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

  This is nondegenerate and holds in particular for all $[x, 0, 0] \in H_0$, which concludes the proof.

$\square$

Remark 4.7.4. Note that $H_{2n}(L;\mathbb{Z}) \cong H_{2n}(M;\mathbb{Z}) \cong \mathbb{Z}$ since M is a closed orientable $2n$-dimensional manifold. This implies by cellular homology that any handle decomposition of L contains a handle of index $\geq 2n > \dim(L)/2 = n+1$, so that by Remark 1.4.20, L cannot carry a Stein structure, and thus by [GP17], L does not admit a Lefschetz fibration.

Remark 4.7.5. From this construction, it follows that the regular fibers $\pi^{-1}(z)$ for $z \neq 0$ can be identified with $M \setminus H$. However, this identification is never symplectic, since the fibers of π have infinite volume with respect to Ω (see [Oba20, Remark 3.7]), whereas $(M \setminus H, \omega|_{M \setminus H})$ has finite volume due to M being a closed symplectic manifold.

Chapter 5

Distinct Strong Symplectic Fillings of the Link of the A_k-Type Singularity

In this chapter, we utilize the machinery of Lefschetz-Bott fibrations through the existence Theorem 4.0.1, together with the strong symplectic filling they induce by Proposition 3.3.1 when restricted to the disk, to exhibit mutually non-homotopic strong symplectic fillings of the link of the A_k-type singularity. The precise statement is the following:

Theorem 5.0.1 ([Oba20, Theorem 1.2])**.** *Let Σ_k be the link of the A_k-type singularity endowed with the canonical contact structure ξ_{can} inherited from the standard contact structure on the unit sphere S^{2n+3}. Then if $\dim(\Sigma_k) \geq 5$, there are at least $\lceil k/2 \rceil + 1$ distinct strong symplectic fillings up to homotopy.*

5.1 Main Ingredients

Let us start by explaining some terminology and collecting the main tools involved in the construction.

Definition 5.1.1. For an integer $k \geq 1$, consider the complex polynomial $f_k \in \mathbb{C}[z_0, \ldots, z_{n+1}]$ given by

$$f(z_0, \ldots, z_{n+1}) = z_0^2 + \ldots + z_n^2 + z_{n+1}^{k+1},$$

and denote its vanishing locus by $V_k := \{\mathbf{z} \in \mathbb{C}^{n+2} \mid f(\mathbf{z}) = 0\}$. This variety has a unique singularity at the origin, which is called the **A_k-type singularity**.

Considering the unit sphere $S^{2n+3} \subset \mathbb{C}^{n+2}$, set

$$\Sigma_k := V_k \cap S^{2n+3},$$

which is known as the **link of the singularity**. We endow Σ_k with the contact structure ξ_{can} inherited from the standard contact structure on the sphere, as defined in Example 1.1.13.

The symplectic fillings of the link of the A_k-type singularity will be obtained by exhibiting different Lefschetz-Bott fibrations from $V_k \cap \mathbb{D}^{2n+4}$ to the disk $\mathbb{D}$ by prescribing their collection of vanishing cycles. The total spaces of these fibrations provide the desired fillings. We may describe these total spaces topologically as arising from a certain gluing construction involving the vanishing cycles:

Construction 5.1.2. Suppose $\pi_i : E_i \to \mathbb{D}$ for $i = 1, 2$ are two topological Lefschetz-Bott fibrations over the disk whose regular fibers are diffeomorphic to F. Fix base points $z_i \in \partial\mathbb{D}$ and denote their fibers by $F_i = \pi_i^{-1}(z_i)$. Now take tubular neighbourhoods $\nu(F_i)$ of the fibers in their respective vertical bundle $\partial_v E_i$. The normal bundle of F_i in $\partial_v E_i$ is trivial, and hence

$$\nu(F_i) \cong [-\epsilon, \epsilon] \times F$$

for some small $\epsilon > 0$. As both fibers F_i are diffeomorphic to F, we may choose a fiber-preserving diffeomorphism

$$f : \nu(F_1) \to \nu(F_2)$$

to glue the total spaces E_i along $\nu(F_i)$, yielding a new manifold

$$E_1 \#_f E_2 := (E_1 \cup E_2)/(x \sim f(x)), \qquad \forall x \in \nu(F_1).$$

This is called the **fiber sum** of E_1 and E_2, and the total space admits a topological Lefschetz-Bott fibration

$$\pi : E_1 \#_f E_2 \to \mathbb{D} \sharp \mathbb{D} \cong \mathbb{D},$$

where $\sharp$ denotes gluing along $\pi_i(\nu(F_i))$, and where π is defined by $\pi|_{E_i} = \pi_i$.

To distinguish the total spaces of the fibrations of interest, we will use the following:

Lemma 5.1.3 ([Oba20, Lemma 4.1])**.** *Let $\pi_i : E_i \to \mathbb{D}$ be topological Lefschetz-Bott fibrations over the disk, for $i = 1, 2$, both with regular fibers isomorphic to F. Let F_i be a regular fiber of π_i, and let $f : F_1 \to F_2$ be a diffeomorphism. Then*

$$\chi(E_1 \#_f E_2) = \chi(E_1) + \chi(E_2) - \chi(F).$$

Proof. Let $p_i : E_i \to E_1 \#_f E_2$ the inclusion $E_i \hookrightarrow E_1 \sqcup E_2$ followed by the quotient projection to $E_1 \#_f E_2$. Then we have $E_1 \#_f E_2 = p_1(E_1) \cup p_2(E_2)$, where $p_1(E_1) \cap p_2(E_2) \cong \nu(F)$. The claim then immediately follows from the fact that $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ for any two subspaces A, B of a topological space so that the interiors of A and B still cover the union $A \cup B$. $\square$

Moreover, to describe the fibrations constructed in the proof, it will be helpful to consider a Lefschetz-Bott fibration over the disk obtained by appropriately restricting the fibration on a complex line bundle over M constructed in Theorem 4.0.1. We refer the reader to appendix A in [Oba20] for details.

Proposition 5.1.4 ([Oba20, Proposition A.2])**.** *Let (L, π, Ω, J, j_0) be the Lefschetz-Bott fibration constructed in Theorem 4.0.1. Then there exist*

- *a compact submanifold with corners $E_c \subset L$ which contains H_0 such that $\pi_c := \pi|_{E_c}$ takes values in $\mathbb{D}^2 \subset \mathbb{C}$, and*
- *a symplectic form $\Omega_c \in \Omega^2(E_c)$ which agrees with Ω on the fibers of π*

such that $(E_c, \pi_c, \Omega_c, J, j_0)$ is a Lefschetz-Bott fibration over the closed unit disk $\mathbb{D}$ whose fibers are canonically identified with $V = M \setminus \mathring{\nu}_M(H)$, and whose critical point set is canonically identified with H. Its monodromy along ∂D is symplectically isotopic to a fibered Dehn twist along the boundary of a regular fiber.

5.2 Construction of the Strong Symplectic Fillings

With these preliminaries in hand, we are ready to prove Theorem 5.0.1.

Outline

Step 1:. We begin in Section 5.3 by considering the compactified cotangent bundle $\mathbb{D}T^*S^n \to S^n$, which will act as the fiber of the symplectic Lefschetz-Bott fibration inducing the desired filling. The first step is to identify $\mathbb{D}T^*S^n$ with a suitable subset of $\mathbb{C}P^{n+1}$, endowed with the Fubini-Study from Examples 1.1.5, to which we can apply Theorem 4.0.1. All symplectic structures that follow will by induced by the Fubini-Study form.

Step 2:. Connecting this to the link Σ_k, we prove in Section 5.4 that the contact manifold $(\Sigma_k, \xi_{\text{can}})$ is contactomorphic to an open book with pages $\mathbb{D}T^*S^n$ and suitable monodromy, which arises through the open book decomposition induced on the boundary of a Lefschetz fibration $V_k \cap \mathbb{D}^{2n+4} \to \mathbb{D}$.

Step 3:. To construct various symplectic fillings of the link, we define different symplectic Lefschetz-Bott fibrations by specifying their fiber $\mathbb{D}T^*S^n$ and vanishing cycles in Section 5.5. The vanishing cycles we consider are $\partial\mathbb{D}T^*S^n$ and the zero section S_0, and we will vary the number of times each manifold occurs as a vanishing cycle to obtain different Lefschetz-Bott fibrations. The fashion in which this is performed gives rise to the same monodromy map for all fibrations - the monodromy of the contact open book identified with Σ_k.

This implies that the open book induced on the boundary of their total spaces is the same as that which was previously identified with Σ_k, so that their total space provides a strong symplectic filling of Σ_k.

Step 4:. The final step is to distinguish these fillings, which is done by comparing their Euler characteristic in Section 5.6. Let E_1 and E_2 be symplectic Lefschetz-Bott fibrations with vanishing cycle $\partial \mathbb{D}T^*S^n$ or S_0, respectively. Note that topologically, the Lefschetz-Bott fibrations constructed above are isomorphic to fiber sums of E_1 and E_2. This is where the identification of $\mathbb{D}T^*S^n$ with a subset of $\mathbb{C}P^{n+1}$ from Step 1 becomes crucial, as Theorem 4.0.1 will allow us to identify E_1 with a better known Lefschetz-Bott fibration on $\mathbb{C}P^{n+1}$ whose Euler characteristic is known. By previous work, we also know the Euler characteristic of E_2. Hence, we may conclude the proof by the repeated application of Lemma 5.1.3.

5.3 Description of the fiber

We describe $\mathbb{D}T^*S^n$ in terms of quadrics in $\mathbb{C}P^{n+1}$. More precisely, set

$$Q^n := \{[z_0 : \ldots : z_{n+1}] \in \mathbb{C}P^{n+1} \mid z_0^2 + \ldots + z_{n+1}^2 = 0\}, \quad Q^{n-1} := Q^n \cap \{z_{n+1} = 0\}.$$

Lemma 5.3.1. *$Q^n \setminus Q^{n-1}$ is diffeomorphic to T^*S^n.*

Proof. Letting $z_{n+1} = 1$, we may identify

$$Q^n \setminus Q^{n-1} \cong \{(w_0, \ldots, w_n) \in \mathbb{C}^{n+1} \mid w_0^2 + \ldots + w_n^2 + 1 = 0\}.$$

In real coordinates $(w_0, \ldots, w_n) = \mathbf{w} = \mathbf{u} + i\mathbf{v}$, we can write this as

$$\{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = -1, \langle \mathbf{u}, \mathbf{v} \rangle = 0\}. \tag{5.1}$$

Note that $\|\mathbf{v}\| \geq 1$. From this set, define a diffeomorphism to T^*S^n by

$$\begin{aligned} \Phi : Q^n \setminus Q^{n-1} &\to T^*S^n \\ \mathbf{w} = \mathbf{u} + i\mathbf{v} &\mapsto \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}, \mathbf{u}\,\|\mathbf{v}\| \right), \end{aligned}$$

where we identify

$$T^*S^n = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|\mathbf{x}\| = 1, \langle \mathbf{x}, \mathbf{y} \rangle = 0\}.$$

The image $\Phi(\mathbf{w})$ is indeed an element of T^*S^n since evidently the first component of $\Phi(\mathbf{w})$ has unit norm and $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ by the description given in Equation (5.1). $\square$

Let $\nu_{Q^n}(Q^{n-1})$ be an open tubular neighbourhood of Q^{n-1} inside Q^n and consider

$$Q := Q^n \setminus \nu_{Q^n}(Q^{n-1}),$$

which can be identified with $\mathbb{D}T^*S^n$. Equip the fiber with $(\Phi^{-1})^*(\omega_{\mathrm{FS}})$, where ω_{FS} denotes the restriction of the Fubini-Study form. Recall that it may be written as

$$\omega_{\mathrm{FS}} = -dd^{\mathbb{C}} \log \left(\sum_{j=0}^{n} |z_j|^2 + 1 \right)$$

on the images of the standard charts. For later use, set

$$\lambda = -d^{\mathbb{C}} \log \left(\sum_{j=0}^{n} |z_j|^2 + 1 \right)$$

and denote by S_0 the zero section of $\mathbb{D}T^*S^n$, which corresponds to $\{\mathbf{u} = \mathbf{0}\}$ in Q.

Also note that $\mathbb{C}P^n$ embedded in $\mathbb{C}P^{n+1}$ as

$$\mathbb{C}P^n = \{[z_0 : \ldots : z_n : 0]\} \subset \mathbb{C}P^{n+1}$$

is a symplectic hypersurface since restricting ω_{FS} gives the Fubini-Study form on $\mathbb{C}P^n$, and we have that $PD[\omega_{FS}/2\pi] = [\mathbb{C}P^n]$. It is hence a Donaldson hypersurface in $(\mathbb{C}P^{n+1}, \omega_{FS})$. Returning to the projective hypersurfaces Q^n and Q^{n-1}, the restriction of ω_{FS} to Q^n is symplectic. Since we may regard

$$Q^{n-1} = \{[z_0 : \ldots : z_n : 0] \in Q^n\},$$

we similarly obtain that Q^{n-1} is a Donaldson hypersurface in $(Q^n, \omega_{\mathrm{FS}})$, and thus we may apply Theorem 4.0.1 for $M = Q^n$ and $H = Q^{n-1}$ to obtain a line bundle over Q^n that admits a symplectic Lefschetz-Bott fibration.

5.4 Open Book Description of Σ_k

We considered in Example 2.1.3 a Lefschetz fibration $V_k(\epsilon) \cap \mathbb{D}^{2n+4} \to \mathbb{C}$ given by the projection to z_{n+1}. Recall that $V_k(\epsilon)$ is the ϵ-level set of the polynomial $f_k = z_0^2 + \ldots + z_n^2 + z_{n+1}^{k+1}$. We saw that this Lefschetz fibration has regular fiber $\mathbb{D}T^*S^n$ and monodromy $\tau_{S_0}^{k+1}$, a product of $k+1$ right-handed Dehn twists along the zero section.

In the case at hand, we consider V_k, the vanishing locus of the polynomial f_k, as the total space, and not its ϵ-level set $V_k(\epsilon)$. One can, however, slightly perturb the polynomial defining V_k in such a way that the projection becomes a Lefschetz fibration with just the same regular fiber and monodromy, and thereby obtain an open book decomposition of the original V_k. This procedure is called *Morsification*, and is described in [KK16, Section 4].

In fact, one can make π into a symplectic Lefschetz fibration by equipping the total space with the 2-form given by

$$d\lambda = -dd^{\mathbb{C}} \log \left(\sum_{j=0}^{n} |z_j|^2 + 1 \right) \Bigg|_{V_k \cap \mathbb{D}^{2n+4}} \in \Omega^2(\mathbb{C}^{n+1}).$$

Note that this is the Fubini-Study form pulled back by a coordinate chart to $\mathbb{C}^{n+1}$, so it is in particular a symplectic form on the entire total space of this Lefschetz fibration.

Hence, the Lefschetz fibration π induces a contact open book decomposition on the boundary $\partial(V_k \cap \mathbb{D}^{2n+4}) = \Sigma_k$ as

$$OB(\mathbb{D}T^*S^n, \lambda; \tau_{S_0}^{k+1}).$$

Thus $(\Sigma_k, \ker(\lambda|_{T\Sigma_k}))$ is contactomorphic to the above open book.

The contact structure ξ_{can} we endow Σ_k with, however, is that induced by restricting the contact structure on S^{2n+3}, which may a priori be different from $\ker \lambda$. However, a computation shows that pulling back λ to S^{2n+3} (by the Hopf map $p : S^{2n+3} \to \mathbb{C}P^{n+1}$) is just the standard Liouville form, so that in fact, $(\Sigma_k, \ker(\lambda|_{T\Sigma_k}))$ is contactomorphic to $(\Sigma, \xi_{\text{can}})$, and hence also to $OB(\mathbb{D}T^*S^n, \lambda; \tau_{S_0}^{k+1})$.

5.5 Exhibiting the Fillings

We specify symplectic Lefschetz-Bott fibrations over $\mathbb{D}^2$ by prescribing their fiber and their vanishing cycles. Denote them by $\pi_\ell : X_\ell \to \mathbb{D}^2$. Let the fiber be $(\mathbb{D}T^*S^n, \lambda)$ for all fibrations, and define the collection of vanishing cycles of π_ℓ to be

$$(\underbrace{\partial\mathbb{D}T^*S^n, \dots, \partial\mathbb{D}T^*S^n}_{\ell}, \underbrace{S_0, \dots, S_0}_{k+1-2\ell}),$$

where ℓ ranges in $0, 1, \dots, \lceil k/2 \rceil$. The monodromy contribution of each $\partial\mathbb{D}T^*S^n$ is a fibered Dehn twist τ_∂ along $\partial\mathbb{D}T^*S^n$ according to Theorem 2.3.9. Note that $\mathbb{D}T^*S^n \cong V_2(1)$ via Φ, and so for the standard symplectic structure on $\mathbb{D}T^*S^n$, the relation by Acu and Avdek stated as Theorem 2.3.8 gives that τ_∂ is symplectically isotopic to $\tau_{S_0}^2$. In fact, this relation also holds for our choice of symplectic structure; see [Oba20, Proposition 3.11]. The contribution to the monodromy by vanishing cycles S_0 is τ_{S_0}, a right-handed Dehn twist along S_0.

Hence the total monodromy is

$$\tau_{S_0}^{2\ell+k+1-2\ell} = \tau_{S_0}^{k+1}.$$

Therefore, each Lefschetz-Bott fibration defined this way induces an open book

$$\partial X_\ell \cong OB(\mathbb{D}T^*S^n, \lambda; \tau_{S_0}^{k+1}),$$

so that since the above open book is contactomorphic to $(\Sigma_k, \xi_{\text{can}})$, each X_ℓ is a strong symplectic filling of $(\Sigma_k, \xi_{\text{can}})$.

5.6 Distinguishing the Fillings

We will do this by computing the Euler characteristic of X_ℓ. We start by considering symplectic Lefschetz-Bott fibrations $\pi_i : E_i \to \mathbb{D}$, for $i = 1, 2$, whose fibers are $(\mathbb{D}T^*S^n, d\lambda)$, where the monodromy of π_1 is τ_∂ (the only vanishing cycle of π_1 is $\partial\mathbb{D}T^*S^n$), and that of π_2 is τ_{S_0} (its only vanishing cycle is S_0).

Consider the space Q^n. We can apply Theorem 4.0.1 for $M = Q^n$ and $H = Q^{n-1}$ to obtain a Lefschetz-Bott fibration on a line bundle over Q^n. Appropriately restricting this fibration, Proposition 5.1.4 yields a Lefschetz-Bott fibration over the disk whose fibers are $Q^n \setminus \nu_{Q^n}(Q^{n-1}) \cong \mathbb{D}T^*S^n$ and whose monodromy is a fibered Dehn twist along the boundary of the fiber. Hence, this Lefschetz-Bott fibration is topologically equivalent to π_1. For fiber bundles with compact base M, fiber F, and total space E, we have $\chi(E) = \chi(F)\chi(M)$, so that

$$\chi(E_1) = \chi(Q^n) = \frac{(-1)^n - 1}{2} + n + 2,$$

where the last equality is due to [Dim92, Exercise 5.3.7 (i)].

As for E_2, it can be shown that E_2 is diffeomorphic to a disk $\mathbb{D}^{2n+2}$ [Oba20, p. 23], so that $\chi(E_2) = 1$.

The total space X_ℓ may now be regarded as the fiber sum of ℓ copies of E_1 and $k + 1 - 2\ell$ copies of E_2, so that by Lemma 5.1.3, we obtain

$$\begin{aligned}
\chi(X_\ell) &= \ell\chi(E_1) + (k + 1 - 2\ell)\chi(E_2) - (k - \ell)\chi(\mathbb{D}T^*S^n) \\
&= \ell\left(\frac{(-1)^n - 1}{2} + n + 2\right) + k + 1 - 2\ell - (k - \ell)\underbrace{\chi(S^n)}_{=1+(-1)^n} \\
&= \ell\left(\frac{(-1)^n - 1}{2} + n + 1 + (-1)^n\right) - k(-1)^n + 1.
\end{aligned}$$

From affine linearity in ℓ of this expression, we see that the X_ℓ are pairwise non-homotopic.

Chapter 6

Outlook

Recall Oba's Proposition 3.3.1 and the associated procedure to obtain a strong symplectic filling of a contact manifold (M, ξ):

1. Find a contact open book description $(M, \xi) \cong OB(F, \lambda; \psi)$;
2. Find Dehn or fibered Dehn twists $\tau_{C_j} \in \mathrm{Symp}(F, d\lambda)$ whose composition is a factorization of the monodromy ψ.

While conceptually simple, the higher-dimensional symplectic mapping class groups are more complicated and not as well-understood, so that factoring the monodromy into fibered Dehn twists is highly nontrivial. Such a factorization may not even exist, of course, seeing that the existence of a Dehn twist in the symplectic mapping class group of $(F, d\lambda)$ presupposes the existence of a Lagrangian sphere in F.

Oba's work, in particular Theorem 4.0.1, provides a potential simplification if one is able to identify the page F with the complement of (a neighbourhood of) a symplectic divisor. In this case, one obtains a symplectic Lefschetz-Bott fibration with regular fiber F whose closed 2-form is globally symplectic (Remark 4.7.1), and appropriately restricting it yields a fibration with monodromy as the fibered Dehn twist along ∂F satisfying the conditions of 3.3.1 to provide a symplectic filling.

One could expect a similar technique as that used for the link Σ_k of the A_k-type singularity, where the page was identified with $F = \mathbb{D}T^*S^n = Q^n \setminus \nu(Q^{n-1})$, to be applicable to other polarized manifolds. Some examples and ways to construct polarized manifolds can be found in [BC01, Section 2.2], the simplest of which is $M = Q^n$ and $H = Q^{n-1}$. The next simplest example is given by

$$(M,\omega) = (\mathbb{C}P^n \times \mathbb{C}P^m, \omega_{\mathrm{FS}} \oplus \omega_{\mathrm{FS}}), \quad H = \left\{ (z,w) \in \mathbb{C}P^n \times \mathbb{C}P^m \mid \sum_{i=0}^{n} z_i w_i = 0 \right\}. \tag{6.1}$$

Of course, one would still need to find a suitable description of $M \setminus H$ along with the knowledge of appropriate mapping class group relations.

Possible references to consult for descriptions of complements of symplectic divisors are [Bir01], [DL19], and [Gir18]. Notably, the polarized symplectic manifold from Equation (6.1) together with the *generalized Lantern relations* have been studied by Torricelli [Tor20] to obtain strong symplectic fillings that are not Stein fillings of the contact manifolds $\mathbb{S}T^*\mathbb{C}P^2$ and $\mathbb{S}T^*\mathbb{R}P^3$.

A Concrete Case

Let us give an outline so as to make the construction from Theorem 4.0.1 more explicit in the simplest case, that is, for

$$M = Q^n = \{[z_0 : \ldots : z_{n+1}] \in \mathbb{C}P^{n+1} \mid z_0^2 + \ldots + z_{n+1}^2 = 0\},$$

equipped with the restriction of the Fubini-Study form ω_{FS}, and the hypersurface inside Q^n given by

$$H = Q^{n-1} = Q^n \cap \{z_{n+1} = 0\}.$$

Theorem 4.0.1 guarantees the existence of a line bundle L with $c_1(L) = -[\omega_{\mathrm{FS}}/2\pi]$ and a Lefschetz-Bott fibration $\pi : L \to \mathbb{C}$.

Recall that the bundle L is defined as

$$L = V \times \mathbb{C} \cup_\Phi (P \times \mathbb{D}(\delta)) \times_\rho \mathbb{C},$$

where

- $P \to H$ is the Boothby-Wang bundle;
- V is the complement of a tubular neighbourhood $\nu(Q^{n-1})$;
- Φ is the gluing map given by

$$\Phi([x,(r_1,\theta_1),(r_2,\theta_2)]) = (\nu([x,(r_1,\theta_1)]),(r_2,\theta_1+\theta_2));$$

- the S^1-actions ρ and $\bar{\rho}$ on $\mathbb{C}$ are defined by

$$\rho(\theta)(z) = e^{2\pi i\theta}(z), \qquad \bar{\rho}(\theta)(z) = e^{-2\pi i\theta}(z).$$

The Boothby-Wang Bundle P

The Boothby-Wang bundle $(P, \alpha) \to (Q^{n-1}, \omega_{\mathrm{FS}}|_{TQ^{n-1}})$ turns out to admit a simple description as the restriction of the Boothby-Wang bundle over $\mathbb{C}P^{n+1}$, which we determine first. Recall the defining property of the Fubini-Study form. Consider the Hopf map p and the inclusion i in the diagram below:

$$\begin{tikzcd} S^{2n+3} \arrow[r, hook, "i"] \arrow[d, "p"] & (\mathbb{C}^{n+2}, \omega_0) \\ (\mathbb{C}P^{n+1}, \omega_{\mathrm{FS}}) & \end{tikzcd}$$

Then the Fubini-Study form satisfies

$$p^*\omega_{\mathrm{FS}} = i^*\omega_0,$$

suggesting that a primitive of the standard symplectic form ω_0 could serve as the connection-contact form of the Boothby-Wang bundle.

Take λ_0 to be twice the primitive of ω_0 inducing the standard contact structure on S^{2n+3}, which in linear coordinates $x_0, y_0, \dots, x_{n+1}, y_{n+1}$ on $\mathbb{C}^{n+2}$ can be written as

$$\lambda_0 = \sum_{i=0}^{n+2} x_i dy_i - y_i dx_i.$$

Recall that the Reeb vector field R_{λ_0} is given by

$$R_{\lambda_0} = \sum_{i=0}^{n+1} x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} = 2 \sum_{i=0}^{n+1} \frac{\partial}{\partial \varphi_i},$$

which corresponds to the infinitesimal generator ξ_1 of the S^1-action on the total space of the Boothby-Wang bundle. The action is hence given by the Reeb flow ϕ_t, which we compute to be

$$\phi_t(\mathbf{z}) = e^{2\pi i t}\mathbf{z},$$

where we consider $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. We conclude that S^1 acts on S^{2n+3} by multiplication in each component, and that the orbits are precisely the fibers of the Hopf map. Hence the S^1-action given by multiplying each entry with $e^{2\pi i\theta}$, $\theta \in S^1$, makes $p : (S^{2n+3}, \lambda_0) \to (\mathbb{C}P^{n+1}, \omega_{\mathrm{FS}})$ into a principal S^1 bundle. The contact form λ_0 is easily checked to be S^1-invariant, and it satisfies $d\lambda_0 = \omega_0 = p^*\omega_{\mathrm{FS}}$. Hence the Boothby-Wang bundle over $\mathbb{C}P^{n+1}$ is given by p.

Therefore, pulling this bundle back to Q^{n-1} gives the Boothby-Wang bundle we are interested in. Explicitly, it is given by

$$p : P = \{(z_0, \dots, z_n, 0) \in S^{2n+3} \mid z_0^2 + \dots z_n^2 = 0\} \longrightarrow Q^{n-1}.$$

Completing the Construction

In principle, it is possible with this description of the Boothby-Wang bundle to carry out the construction of L step by step to obtain an explicit bundle. For example, the neighbourhood of H, denoted by $\nu_{Q^n}(H)$, is symplectomorphic to $P \times_{\bar{\rho}} \mathbb{D}(\delta)$. Given that S^1 acts on $P \subset S^{2n+3}$ by multiplication, which is just the action defining $\mathbb{C}P^{n+1}$, one can obtain that

$$\nu_{Q^n}(H) \cong Q^{n-1} \times \mathbb{D}(\delta)/([\mathbf{z}], w) \sim ([\mathbf{z}], e^{2\pi i\theta}w), \quad \theta \in S^1.$$

One can continue along these lines to obtain expressions for the other spaces involved, namely

- the bundle over $\nu_{Q^n}(H)$, given by $(P \times \mathbb{D}(\delta)) \times_\rho \mathbb{C}$;
- the bundle over V, given as the trivial bundle $V \times \mathbb{C}$;
- the glued space $L = V \times \mathbb{C} \cup_\Phi (P \times \mathbb{D}(\delta)) \times_\rho \mathbb{C}$;
- the Lefschetz-Bott fibration $\pi : L \to \mathbb{C}$, defined on $V \times \mathbb{C}$ by the projection to the $\mathbb{C}$-factor, and on the bundle over $\nu_{Q^n}(H)$ by multiplication in the $\mathbb{D}(\delta)$- and $\mathbb{C}$-factors, interpolating with a smooth function μ as in Section 4.5 so as to make it smooth.

However, writing out the definitions of these spaces does not immediately lead to a deeper understanding of the constructed Lefschetz-Bott fibration. The complement V still warrants a more rigorous description, as well as the spaces involved after all identifications have been made.

In the end, a dream result would be a tractable identification of the bundle L with the tautological line bundle $\mathcal{O}(-1)$, to which it is isomorphic due to them having the same first Chern class.

Bibliography

[AA16] B. Acu and R. Avdek, "Symplectic mapping class group relations generalizing the chain relation," *International Journal of Mathematics*, vol. 27, no. 12, 2016.

[Ale23] J. W. Alexander, "A lemma on systems of knotted curves," *Proceedings of the National Academy of Sciences*, vol. 9, no. 3, pp. 93–95, 1923.

[Arn95] V. I. Arnol'd, "Some remarks on symplectic monodromy of Milnor fibrations," in *The Floer memorial volume*, ser. Progress in Mathematics, H. Hofer, C. H. Taubes, A. Weinstein, and E. Zehnder, Eds., vol. 133, Birkhäuser, Basel, 1995, pp. 99–103.

[Arn+98] V. I. Arnol'd, V. V. Goryunov, O. V. Lyashko, and V. A. Vasil'ev, "Singularity Theory I." Springer Verlag Berlin, 1998, Translated from the 1988 Russian original by A. Iacob.

[AGV88] V. I. Arnol'd, S. M. Gusein-Zade, and A. N. Varchenko, "Singularities of Differentiable Maps. Volume 2, Monodromy and Asymptotics of Integrals." Birkhäuser, 1988.

[BC01] P. Biran and K. Cieliebak, "Symplectic topology on subcritical manifolds," *Commentarii Mathematici Helvetici*, vol. 76, pp. 712–753, 2001.

[Bir01] P. Biran, "Lagrangian barriers and symplectic embeddings," *GAFA, Geometric Functional Analysis*, vol. 11, pp. 407–464, 2001.

[BW58] W. M. Boothby and H. C. Wang, "On contact manifolds," *Annals of Mathematics*, vol. 68, no. 3, pp. 721–734, 1958.

[Can06] A. Cannas da Silva, "Lectures on Symplectic Geometry." Springer-Verlag Berlin Heidelberg, 2006.

[CDK14] R. Chiang, F. Ding, and O. V. Koert, "Open books for Boothby-Wang bundles, fibered Dehn twists and the mean Euler characteristic," *Journal of Symplectic Geometry*, vol. 12, no. 2, pp. 379–426, 2014.

[CE10] K. Cieliebak and Y. Eliashberg, "From Stein to Weinstein and Back." American Mathematical Society Colloquium Publications, 2010.

[Dim92] A. Dimca, "Singularities and Topology of Hypersurfaces." New York: Springer, 1992.

[DL19] L. Diogo and S. Lisi, "Symplectic homology of complements of smooth divisors," *Journal of Topology*, vol. 12, no. 3, pp. 967–1030, 2019.

[Don96] S. K. Donaldson, "Symplectic submanifolds and almost-complex geometry," *Journal of Differential Geometry*, vol. 44, no. 4, pp. 666–705, 1996.

[Don99] S. K. Donaldson, "Lefschetz pencils on symplectic manifolds," *Journal of Differential Geometry*, vol. 53, no. 2, pp. 205–236, 1999.

[Eli91] Y. Eliashberg, "Filling by holomorphic discs and its applications," in *Geometry of Low-Dimensional Manifolds: Symplectic Manifolds and Jones-Witten Theory*, S. K. Donaldson and C. B. Thomas, Eds. Cambridge University Press, 1991, vol. 2, pp. 45–68.

[Etn05] J. B. Etnyre, "Lectures on open book decompositions and contact structures," 2005. arXiv: math/0409402 [math.SG].

[Gei08] H. Geiges, "An Introduction to Contact Topology." Cambridge University Press, 2008.

[Gir02] E. Giroux, "Géometrie de contact: de la dimension trois vers les dimensions supérieures," in *Proceedings of the International Congress of Mathematicians*, (Beijing), T. Li, Ed., vol. 2, Higher Education Press of China, 2002, pp. 405–414.

[GM] E. Giroux and J. P. Mohsen, "Structures de contact et fibrations symplectiques au dessus du cercle," unpublished manuscript.

[GP17] E. Giroux and J. Pardon, "Existence of Lefschetz fibrations on Stein and Weinstein domains," *Geometric Topology*, vol. 21, no. 2, pp. 963–997, 2017.

[Gir18] E. Giroux, "Remarks on Donaldson's symplectic submanifolds," 2018. arXiv: 1803.05929 [math.SG].

[Gir] E. Giroux, "Contact geometry: From dimension three to higher dimensions," Translated from the French original by B. Acu. [Online]. Available: https://people.math.ethz.ch/~bacubulut/assets/acu_translation.pdf.

[Gom98] R. Gompf, "Handlebody construction of Stein surfaces," *Annals of Mathematics*, vol. 148, no. 2, pp. 619–693, 1998.

[GS99] R. Gompf and A. Stipsicz, "4-Manifolds and Kirby Calculus." American Mathematical Society, 1999.

[Gro85] M. Gromov, "Pseudo holomorphic curves in symplectic manifolds," *Inventiones Mathematicae*, pp. 307–347, 1985.

[Hop27] H. Hopf, "Vektorfelder in n-dimensionalen Mannigfaltigkeiten," *Mathematische Annalen*, vol. 96, pp. 225–249, 1927.

[Hus94] D. Husemöller, "Fibre Bundles." Springer-Verlag New York, 1994.

[KK16] M. Kwon and O. van Koert, "Brieskorn manifolds in contact topology," *Bulletin of the London Mathematical Society*, vol. 48, no. 2, pp. 173–241, 2016.

[Law78] T. Lawson, "Open book decompositions for odd dimensional manifolds," *Topology*, vol. 17, pp. 198–192, 1978.

[LHW18] S. Lisi, J. V. Horn-Morris, and C. Wendl, "On symplectic fillings of spinal open book decompositions I: Geometric constructions," 2018. arXiv: 1810.12017 [math.SG].

[MNW13] P. Massot, K. Niederkrüger, and C. Wendl, "Weak and strong fillability of higher dimensional contact manifolds," *Inventiones Mathematicae*, vol. 192, no. 2, pp. 287–373, 2013.

[MS74] J. W. Milnor and J. D. Stasheff, "Characteristic Classes." Princeton University Press, 1974.

[Mil68] J. Milnor, "Singular points of complex hypersurfaces," ser. Annals of Mathematics Studies. Princeton University Press, 1968, vol. 61.

[Oba18] T. Oba, "Higher dimensional contact manifolds with infinitely many Stein fillings," *Transactions of the American Mathematical Society*, vol. 370, no. 7, pp. 5033–5050, 2018.

[Oba20] T. Oba, "Lefschetz-Bott fibrations on line bundles over symplectic manifolds," *International Mathematics Research Notices*, vol. 00, no. 0, pp. 1–40, 2020.

[ÖS04a] B. Özbağcı and A. Stipsicz, "Contact 3-manifolds with infinitely many Stein fillings," *Proceedings of the American Mathematical Society*, vol. 132, no. 5, pp. 1549–1558, 2004.

[ÖS04b] B. Özbağcı and A. Stipsicz, "Surgery on Contact 3-Manifolds and Stein Surfaces." Springer-Verlag Berlin Heidelberg, 2004.

[Per07] T. Perutz, "Lagrangian matching invariants for fibred four-manifolds: I," *Geometry and Topology*, vol. 11, pp. 759–828, 2007.

[Qui79] F. Quinn, "Open book decompositions, and the bordism of automorphisms," *Topology*, vol. 18, pp. 55–73, 1979.

[RS18] J. Robbin and D. Salamon, "Introduction to differential topology," 2018. [Online]. Available: https://people.math.ethz.ch/~salamon/PREPRINTS/difftop.pdf.

[Sei97] P. Seidel, "Floer homology and the symplectic isotopy problem," Ph.D. dissertation, The University of Oxford, 1997.

[Sei99] P. Seidel, "Lagrangian two-spheres can be symplectically knotted," *Journal of Differential Geometry*, vol. 52, no. 1, pp. 145–171, 1999.

[Sei03] P. Seidel, "A long exact sequence for symplectic floer cohomology," *Topology*, 2003.

[Sei08] P. Seidel, "Fukaya Categories and Picard-Lefschetz Theory," ser. Zürich lectures in advanced mathematics. European Mathematical Society, 2008.

[Tam72] I. Tamura, "Spinnable structures on differentiable manifolds," *Proceedings of the Japan Academy*, vol. 48, pp. 293–296, 1972.

[Tam73] I. Tamura, "Foliations and spinnable structures on manifolds," *Annales de l'Institut Fourier*, vol. 23, no. 2, pp. 197–214, 1973.

[Tam76] I. Tamura, "Topology of Foliations: An Introduction." American Mathematical Society, 1976, Translated from the 1976 Japanese original by K. Hudson.

[TW75] W. P. Thurston and H. E. Winkelnkemper, "On the existence of contact forms," *Proceedings of the American Mathematical Society*, vol. 52, no. 1, pp. 345–347, 1975.

[Tor09] D. M. Torres, "Open book decompositions and contact geometry," 2009. [Online]. Available: https://www.math.tecnico.ulisboa.pt/~martinez/files/obd.

[Tor20] B. C. Torricelli, "Model projective twists and generalised lantern relations," 2020. arXiv: 2008.02758 [math.SG].

[Waj99] B. Wajnryb, "An elementary approach to the mapping class group of a surface," *Geometry and Topology*, vol. 3, pp. 405–466, 1999.

[WW16] K. Wehrheim and C. T. Woodward, "Exact triangle for fibered dehn twists," *Research in the Mathematical Sciences*, vol. 3, no. 17, 2016.

[Wei71] A. Weinstein, "Symplectic manifolds and their lagrangian submanifolds," *Advances in Mathematics*, vol. 6, no. 3, pp. 329–346, 1971.

[Wei91] A. Weinstein, "Contact surgery and symplectic handlebodies," *Hokkaido Mathematical Journal*, vol. 20, pp. 241–251, 1991.

[Win73] H. E. Winkelnkemper, "Manifolds as open books," *Bulletin of the American Mathematical Society*, vol. 79, no. 1, 1973.

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